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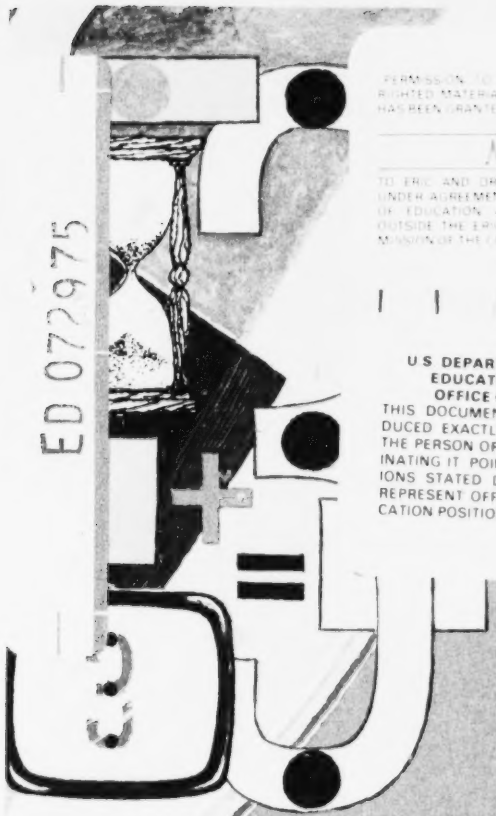
ABSTRACT

In this 1967 booklet, influences of technology, the non-achiever and the culturally disadvantaged, and the revolt against formalism are discussed in relation to the modern mathematics curriculum. Some projects and school programs described include PLATO, the Nuffield Project, the Nova School Program, Advanced Placement Program, and teacher education programs. Materials (such as mirror cards, attribute blocks, and textbooks), methods (CAI, math labs, discovery learning), curriculum (new topics and courses, general math), and evaluation also are discussed. A bibliography is included. (DT)



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The Changing Curriculum: MATHEMATICS

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The Changing Curriculum: MATHEMATICS

Prepared for the ASCD
Commission on Current Curriculum Developments

by

Robert B. Davis

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Foreword

The Changing Curriculum: Mathematics follows *The Changing Curriculum: Science* as the second in a series of booklets growing out of the work of the ASCD Commission on Current Curriculum Developments.

In keeping with the intent of the Commission, Robert B. Davis has aimed this booklet at the growing, evolving forefront of the mathematics curriculum. It is as new, current and sparkling as the latest model automobile. In fact, portions of it anticipate developments in the field. Yet these new developments in computer-assisted instruction, concept development and sequence are seen in a perspective reaching back to Euclid and ahead to a New Progressive Education.

The reader will learn much from this booklet about what is now happening in mathematics education. Perhaps more importantly, however, he will see these developments in the light of the latest technology affecting all curriculum areas and nearly all curriculum decisions.

May 1967

J. HARLAN SHORES
President, 1967-68
Association for Supervision and Curriculum Development

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Final editing of the manuscript and production of this booklet were the responsibility of Robert R. Leeper, Associate Secretary and Editor, ASCD Publications. Technical production was handled by Mary Ann Lurch, assisted by Sharon L. Landau, Marcia Abercrombie and Teola T. Jones.

The Second Decade of "Modern Mathematics Curricula"

Despite newspaper reports, publishers' claims, cartoons, and even songs to the contrary, the "new mathematics revolution" has not taken place, but—considering the pressures that are building up—it probably will, possibly within the next ten years, provided our society is not torn apart by World War III or something of that sort. What is happening bears more than a small resemblance to most political revolutions, especially in the feeling one gets of "accumulating forces" and "impending breaks in the existing structure." Real revolutions have an avalanche-like quality, in that those who seem to be at the forefront soon find themselves running as fast as they can to keep from being buried. The mathematics revolution assumes more of this imminent-avalanche flavor every day. One of these days it may happen. Or—more accurately—one of these days it may really begin, for the one sure fact that stands out is that the real revolution, if it does begin, will not be finished in a day, or a week, or a year.

For professional educators the situation is not necessarily either "good" or "bad." It is one more input to be fed into their decision-making apparatus, and to be coped with as wisely and objectively as possible. Where the "revolution" can be made to yield up some advantage, it presumably will be. Where it confronts operating systems with additional and heavy new burdens, these can be borne in the best way possible. This is what professionalism, at its best, means in any field of endeavor.

As we look at the forces and pressures in more detail, we can abandon objectivity at least for a moment and be humanly grateful for the relatively close rapport that presently exists between the various sectors of the academic community. Today there are more school administrators and historians of education on a first-name basis with more mathematicians and physicists than at any time since at least the 1930's. This much bodes well for the strength and wisdom with which we can respond to the avalanche when it really arrives.

Before going further, the reader may legitimately wonder what direction we are about to take. After considering various matters as briefly

as possible, but deeply enough to establish some communication, we shall want to draw some conclusions. The author did not draw his conclusions in advance, and they came as a surprise to him. It may be well to reorganize this report, in order to give the reader more advance warning: after extensive study of projects and schools during the past year, these conclusions seem to emerge with considerable force:

1. The "problems" or "pressures" underlying the present changes are very real and very serious. They include a technology which is becoming more complicated more rapidly than almost anyone seems to realize. What "advanced" universities and engineering schools taught 20 years ago often seems refreshingly quaint—something like hand-loomed textiles—when seen against today's sophistication and complexity. They include also the multiple crises of our largest cities, in which the future of our society and the future of our education are inextricably intertwined. (Although we shall not discuss this aspect here, they include also the education of world citizens who can and do think in terms of the human race.)

2. The "new mathematics" thus far has constituted an entirely inadequate response to these needs. On the one hand, far too little has actually been accomplished; on the other hand, the thinking *about* school mathematics has nearly always been shallow and superficial—it is time that we resolve to seek much deeper discussions of what schools, mathematics and children have to do with one another.

What Are the Underlying Pressures?

There have been many discussions of the forces that underlie the "mathematics revolution," but a large proportion of these have shot wide of the mark. As John Goodlad has pointed out,[1] it was not Sputnik that started it all, and as Bernard Asbell has pointed out,[2] we are not mainly confronted by the aspects of unemployment caused by automation. One could come a great deal closer by reading Cremin's book, *The Transformation of the School*,[3] and by visiting PLATO at the University of Illinois.

It seems imperative at this stage in the "curriculum evolution" movement that we focus some of our attention on the broad realities that will influence educational operations in the next decade or two.

What we shall *not* do here is list and describe the various "new mathematics" projects. Several lists and descriptions already exist and are readily available,[4] and one more would add nothing. We can, however, follow the suggestion above, and begin with a visit to the PLATO Project at the University of Illinois (Urbana, Illinois).

The Two-pronged Influence of Technology

The PLATO Project is embedded in, and related to, an abundant proliferation of acronymic organizations that grew out of the scientific effort of World War II. As we shall see, the cooperation among these several organizations greatly enhances the strength of each of them. To begin with, there is the Coordinated Science Laboratory, primarily a matter of physics and electrical engineering, that has given birth to some almost

[1] In: John Goodlad, Renata Von Stoephasius and M. Frances Klein. *The Changing School Curriculum*. New York: The Fund for the Advancement of Education, 477 Madison Avenue, 1966.

[2] In: Bernard Asbell. *The New Improved American*. New York: McGraw-Hill Book Company, Inc., 1965.

[3] Lawrence A. Cremin. *The Transformation of the School*. New York: Vintage Books (Random House, Inc.), 1961.

[4] Cf. Robert B. Davis. "Mathematics." Chapter 6 in *New Curriculum Developments*. Glenys Unruh, editor. Washington, D.C.: Association for Supervision and Curriculum Development, 1201 Sixteenth Street, N.W., 1965; and also J. David Lockard. *Report of the International Clearinghouse on Science and Mathematics Curricular Developments 1966*. College Park, Maryland: Science Teaching Center, University of Maryland.

incredibly powerful and flexible digital computing equipment and "teaching machine" hardware.

In addition, there is Max Beberman's UICSM Project ("University of Illinois Committee on School Mathematics"), which produced one of the earliest "new math curricula" for bright children in grades 9-12, and has recently undertaken the construction of an entirely new mathematics curriculum for low-achievers, beginning in grade 7; there is PLATO itself ("Programmed Logic for Automatic Teaching Operations"); there is CIRCE ("Center for Instructional Research and Curriculum Evaluation"); there is "Uni High"—the University of Illinois' excellent laboratory school; there is the School Science Curriculum Project, Richard Salinger's project in elementary school science; there is CERL ("Computer-Based Education and Research Laboratory"), headed by Louis Volpp, an economist turned educational administrator; and there is USOE-supported Project SIRA ("System for Instructional Response Analysis"), headed by Jack Easley.

PLATO, CERL, *et al.*, contain a major clue to why there soon will be a mathematics revolution. These organizations personify the two-pronged impact of modern electronic technology: in the first place, the *products produced* by this technology will inevitably enter both home and classroom, and will change drastically the process of learning (along with quite a bit else, in fact); in the second place, a technology as sophisticated as this—involving plasma physics, electronic information filters, and micro-second computer memory units that can be read visually at a glance by a human being!—bears very little resemblance to the technology of 1876 or 1910. Hence this technology requires entirely different educational programs to turn out a different kind of educated adult.

I have picked the dates 1876 and 1910 with malice aforethought, since I wish to argue that what we are witnessing in the 1960's is probably best described as a rebirth (or perhaps an extension) of progressive education, and that it is no exaggeration to say that the underlying forces which confront us today are indicated more clearly in Cremin's book, *The Transformation of the School*, than in any other volume the reader is likely to encounter.

The year 1876 witnessed the Philadelphia Centennial Exposition, which introduced into the United States the *instruction shops* for carpentry, blacksmithing, etc., that had been organized by the Russian educator, Victor Della Vos, for the Moscow Imperial Technical School. Reading Cremin's discussion of Della Vos' work, and its reception in the United States, gives one the feeling that many important parallels with 1967 are not hard to discern:

> The Exposition had boasted literally hundreds of displays. . . . It is not

surprising, though, that pedagogical innovations associated rather directly with industrial prosperity had come under the closest scrutiny. In the end, a few displays of tools from Moscow and St. Petersburg literally stole the show; for these objects showed the West for the first time that Russian educators had finally scored a breakthrough on the thorny problem of how to organize meaningful, instructive shop training as an essential adjunct of technical education.

. . . When the [Moscow Imperial Technical] School had been created by royal decree in the spring of 1868, the effort had been to complement the work in mathematics, physics, and engineering with on-the-job training in . . . shops built expressly for teaching purposes.

. . . It is said that President John R. Runkle of the Massachusetts Institute of Technology was strolling through Machinery Hall one day when he happened upon the Russian display cases. American education was never the same thereafter. Runkle had been wrestling with the shop problem at M.I.T. and for him the Russian solution held "the philosophical key to all industrial education."[5]

We need not be concerned here with the precise form of "instruction in the manual arts" that was involved, except to note two facts: first, under the leadership of Runkle and of Calvin M. Woodward of Washington University in St. Louis, a revolution in American schools *was* effected, and one of the threads of progressive education began to enter the fabric of American life, at the school level quite as much as at the college level. Second, the technical education involved had dealt with *joinery, carpentry, blacksmithing,* and similar matters. That is a very far cry, indeed, from ionic propulsion sytems for spacecraft, from nitrogen-gas computer memories, or from titanium Mach 3 aircraft. The technology of 1967 is by no means based on carpentry and blacksmithing. There *has* to be some impact of today's technology on our schools, and for the most part it has not yet occurred—notwithstanding Jay Samiljan's teaching TV studio in George Westinghouse Vocational High School in Brooklyn, or the relatively few courses offered around the country that relate to digital computers.

The other date—1910—was chosen as that of some educational writing by Henry Wallace, the editor of *Wallace's Farmer*. The following description by Cremin indicates rather clearly the relevance of the agricultural education movement—another major thread in the fabric of progressive education—to the situation that is nowadays reflected in the projects in mathematics and science:

Wallace agreed . . . on abandoning the "cut-and-dried formula of a period when a man was 'educated' only when he knew Greek and Latin," and suggested that there be less adherence to textbooks, more concern with the all-around development of children, and unceasing attention to the rudiments of

[5]Cremin, *op. cit.*, pp. 24-25.

agriculture. "It is hard," he wrote, "for many a middle-aged farmer to get a clear idea of what is meant by protein, carbohydrates, nitrogen-free extract, etc. Now, these terms are no harder than many which the pupils learn and which are of no earthly use to them in their everyday lives." The teachers' guides should come not from high schools, normal schools, or colleges, but from farmers themselves, who know best what their children need. Instead of depending on textbooks, teachers should experiment in the classroom with seeds, with the Babcock milk tester, with honeycombs, or with any other practical material. . . .[6]

Wallace's "middle-aged farmer" struggling with the idea of nitrogen-free extract surely resembles many a middle-aged mid-twentieth-century resident struggling with "the Cartesian product of two sets," or with the axioms for an ordered field. Where the resemblance completely fails, however, is between the technology of the agrarian-industrial nation in 1910, and our aero-space and electronics technology today. Progressive education was fed by many streams, including those that sprang from farmers and industry fighting against an academic establishment that dealt in Greek and Latin, in an attempt to convert it to carpentry, blacksmithing, and milk testing. The present curriculum evolution movement is in part a matter of modern science and technology trying to fight against an academic establishment that seems devoted to carpentry, a bland and inaccurate version of pre-Keynesian economics, the mastery of ancient arithmetic algorithms and other antiquated computational devices, and Scott's *Lady of the Lake*, in an effort to convert it to programs having more relevance to modern technology and mid-twentieth-century styles of living.

I have argued that PLATO shows us two powerful forces at work: a technology with a voracious appetite for a new kind of "educated man," and an array of technological capabilities that must inevitably enter the classroom and transform it unrecognizably.

Both propositions are present in the person and work of Donald Bitzer, an electrical engineer who is director of Project PLATO. Bitzer has developed a display device to replace TV screens in teaching-machine terminals, and is now planning the assembly of a $3,000,000 central computer, with perhaps 6,000 terminals at the University of Illinois, in nearby colleges, in schools, offices and even homes.[7] Using his new display device (which looks generally like a TV screen but is far cheaper, far more

[6] Cremin, *op. cit.*, p. 44. One could easily rewrite this passage to describe the schools in Oak Ridge or Huntsville or Cape Kennedy today, the "middle-aged farmer" being replaced by a "middle-aged computer specialist" or a "middle-aged space scientist."

[7] That is to say, using "time sharing," 6,000 students could be using this single computer at the same time, each student having a complete "teaching machine" terminal at his own desk. Present installations rarely run beyond 20 terminals or so.

reliable, and absolutely free of distortion), combined with a teletype machine (of an improved type that Bitzer is now working on), and using computer programs ("software") already developed, Bitzer can give each prospective author a complete computer terminal *in his own home, operating over his existing telephone line.*

With this, the author can design (say) a ninth-grade algebra "course" at the terminal in his home, store it in the computer (which will, of course, help the author to write the material in the first place), and have students work through the course on the computer terminals at their desks in school. The author can query the computer on any details of student performance at any time, and can use the reported results to modify the course however he wishes, with the computer inserting the modifications immediately in all the proper places. There may be no recognizable "textbook" or "written test" used at any point! The instruction can be entirely individualized, with no two students pursuing identical paths through the material—and the cost will be about 10 cents per hour!

What is even more, all of this hardware (in smaller quantity) and much of this software already exist, and the University has already accumulated some thousands of student hours of study in this way.

This is a technology that will not be denied. This is no "programmed learning" or "teaching machine" fad that will soon pass. These devices are far more powerful and far more flexible than either traditional teaching machines or educational television, and they are far cheaper. Every response of every student can be recorded and studied, every student can pursue his own program of study independently of all other students (but profiting from the experience which the computer and the author have accumulated from other students), and the total cost is about 10 cents per hour per student. This is equipment as simple, as reliable, and as revolutionary as your ordinary home telephone.

Here is a whole new job category: to be a PLATO course author. Here is an entirely new program for doctoral studies: to prepare to be a PLATO course author.

It is no wonder that major United States industries are hastening to buy up publishers and other education-related companies: Columbia Broadcasting System has acquired Creative Playthings and a major interest in Holt, Rinehart, and Winston; the National Broadcasting Company-RCA have merged with Random House and Alfred A. Knopf; Litton Industries has acquired American Book Company; Time, Inc., and General Electric have purchased Silver Burdett Company and formed General Learning Corporation; Minnesota Mining and Manufacturing has joined Newsweek in an educational venture; Sylvania and Reader's Digest have a similar

joint venture; Raytheon has purchased four education firms, including D.C. Heath; and Xerox Corporation has created an education division, combining several firms it has purchased (University Microfilms, Basic Systems, and American Education Publications, publishers of *My Weekly Reader*). International Business Machines (IBM) has purchased Science Research Associates, an educational publisher of texts and tests.[8] Even this impressive list is incomplete.

As Leander Smith, the former philosopher and mathematics teacher who serves IBM-SRA as an "advance planner" for their educational division puts it:

> I want to say to teachers that this is not something that will soon pass over. If teachers become actively involved early enough, they can influence CAL ["computer-assisted learning"] so that it will offer them greater freedom, greater flexibility, and greater effectiveness—as well as a pleasanter professional situation in general. But if teachers ignore CAL, they will probably find that it has grown in a fashion that will produce more rigid and narrower constraints, and less opportunity for flexibility and change.

I want to comment again on the two-sided nature of this kind of technology. We are no longer educating mainly farmers, blacksmiths and carpenters. We must now educate men like Donald Bitzer and Leander Smith, and all of the other technical and software people who will be involved in tomorrow's technology and tomorrow's education. If some echo of *Wallace's Farmer* sounds from the wings . . . is that surprising? The graduates of our schools must be a new kind of educated man, not brought up on "textbooks and Latin and Greek," nor yet on "honeycombs and the Babcock milk tester." They must be educated for today and tomorrow, and tomorrow begins at midnight tonight.

On the other hand, no one who watches students sitting in a booth with a TV screen and an electric typewriter connected to Bitzer's computer (which even trades jokes with the students), and who studies the highly feasible economics of Bitzer's proposed 6,000 such booths, will be inclined to deny that this equipment will soon find its way into our classrooms, and *this* equipment will make a difference. Sixteen-mm movie projectors are in our schools, but they really have not made much difference. Overhead projectors are in a few of our schools, but they really have not made much difference. There are a few desk calculators available to students, and they make a difference if they are well used. Yet none of this has prepared us for some technology that will really make a difference—and precisely such technology does now exist!

[8] "Big Corporations Increasing Interest in Education Field." *The St. Louis Post-Dispatch*, October 9, 1966. p. 4 G.

To see why Leander Smith warns that the future can bring several different situations, not all equally desirable, we need to consider some of the kinds of things that are presently being done with computer-assisted learning.

In the first place, CAL equipment is sometimes used in a "tutorial mode," where the machine sets the task and the student responds. Professor Patrick Suppes, at Stanford University, has an entire elementary school arithmetic drill program now in operation that uses a computer in this fashion, as well as a program in logical proof-making for second graders.

This is not, however, anything like the outer boundary of what is possible—or even of what is already in use. Computers can also be used in an "inquiry mode." For example, the student in one program is told that he works for an automobile company that purchases accelerator springs from three different suppliers. There have been reports lately of defective springs. The student is assigned to "study the situation and correct it." Now, the *student* asks questions, and the *computer answers*. The student may request a sample spring from Company A. (A picture of this sample spring now appears on his TV screen.) He may ask the computer to hang a five-pound weight on the spring. (The spring on the TV screen stretches an appropriate amount, and the measurement of the amount of stretching is indicated in inches.) The computer's responses include experimental error (if the student repeats, and again hangs a five-pound weight on the spring, the computer may—or may not—indicate the same amount of stretching). If the student hangs too heavy a weight on the spring, he will exceed the elastic limit of the spring in a realistic fashion.

Of course, there *is no real spring*. This is all being done by computer simulation, based on numerical data previously obtained from real springs.

If the student does "ruin" a "spring" by exceeding the elastic limit, before he has completed sufficient testing, he can request the computer to supply him with another spring from Company A (and a new spring will appear on the TV screen). Perhaps this time the student will be more careful in selecting the "weights" to "hang" on the "spring."

Every response of the student is being recorded. When he has identified the offending supplier or suppliers, he writes them reports on official company stationery, indicating the violations of the acceptability criteria, and terminating the order.

The example we have just seen is perhaps a combination of "inquiry mode" with "simulation." Both are presently in use in various programs, either separately or in combination.

In addition, there are modes of computer use where the computer supplies hints and other aids upon request (for example, it displays a graph of data already accumulated), where the computer is used in combination with a human teacher (something like Dr. Spock's book saying "See your doctor!"), or where the computer is used in conjunction with manipulatable physical materials or in conjunction with an actual laboratory with *real* springs, or whatever.

In the PLATO project, after enough thousands of hours of data on real live students has been accumulated—and this point is being approached right now—it is of course possible to use this data (properly analyzed) to allow the computer to *simulate* students. Authors can then try out a new course—again without leaving their homes—on *simulated* students, and carry the course through several revisions before it is presented to any live students.

University students can gain some experience in teaching simulated students (Bert Kersh of the Oregon State Department of Education has some equipment to do this at present, although it is vastly simpler and necessarily more limited than the PLATO facilities), before they come into contact with living students.[9]

There even exist computer programs for generating original proofs of mathematical theorems, and programs for checking the correctness of proofs of theorems. Bitzer suggests—half in jest, but from similar jests atomic physics was born—that a computer making up new proofs be connected to another computer checking proofs for correctness, with the prospect of producing original mathematical research untouched by human hand.

Where reality and feasibility end, and fantasy and science fiction begin, is by no means easy to say. Engineers at Stanford University expect within a few years to have computers that can understand the human voice, so that very young children will merely talk to the machine, without the burden of reading, writing, or selecting the proper button to push. Although the ability to listen and to "understand" lies at least a few years in the future, computers are already able to *talk* to people, and the reader may even have listened to a computer talking to *him* (for instance, if he has dialed a telephone number that is no longer in use).

Computers, however, differ greatly among themselves, and the kind of use that is envisioned will determine the kinds of computers that will be designed, built, and—here's the rub—purchased. This, in turn, will be determined by our views of a desirable education. If we are wrong

[9]It should be added that some observers question both the value and the feasibility of using "simulated students" in this way. It is nonetheless relevant that such matters are being discussed.

about this—and there are several quite different schools of thought—then we shall find ourselves owning and necessarily using the wrong kind of equipment, both in terms of hardware and in terms of software. The situation is quite similar to the question of large group and small group instruction; if we plan for this in advance, then we design and erect a building with large lecture halls, with small seminar rooms, and perhaps with movable partitions. If we do not, we probably will find ourselves owning, and necessarily using, a building in which every room holds exactly thirty children.

Computers are not the whole of modern technology. What is almost the opposite extreme in educational practice was operated during the summer of 1966 by the Madison Project, of Syracuse University and Webster College. A team of about 40 teachers experienced in the use of the Madison Project approach to the teaching of modern elementary school mathematics flew, by jet airplane, into Los Angeles, from there to Chicago, from there to San Diego, and from there to New York City, conducting a one or two week workshop for several hundred teachers in each city, using, among other things, closed circuit TV. The Madison Project has always "spread the word" mainly by direct face-to-face contact among people; its present ability to do this effectively on a large scale depends entirely upon modern jet aircraft, and incidentally also uses TV, videotape, 16 mm film, and 8 mm film cartridges. The Project also uses "math labs," and seminar-type discussion groups. *It makes relatively little use of printed books.*

Computerized courses represent, in a sense, the opposite path of escape from textbooks: if the Madison Project finds books "too mechanical" and prefers an interchange between human beings, computerized instruction finds books "too technologically primitive," and prefers a more individualized approach via a computer that interacts with each student separately.

Both seek to escape the rigidities and unhappy compromises that seem to be intrinsic to the nature of textbooks. However, small-group instruction, seminar-type discussions, and a student-centered school atmosphere are *not* incompatible, by any means, with the use of computer-assisted learning devices. As Professor Suppes points out, they are in fact complementary, and directed toward basically similar goals—such as rescuing the individual child from the anonymity of being one of 30 children to whom the teacher is talking, or one of 100,000 children for whom a textbook was written. Both are concerned with methods for revising the curriculum much more often—at least as often as once a year, say—than was possible during the textbook-using period of American educational history. (Rescuing the child and the curriculum from domina-

tion by the textbook has been a time-honored goal of progressive education, and is an entirely reasonable and proper goal, even today.)[10]

The Non-achiever, the School Dropout, and the Culturally Disadvantaged

Obviously, another source of pressure on the curriculum comes from the increased recognition that far too many children get off to an impossibly bad start in life. We shall not discuss this as a separate issue, because the need here—for a more child-centered school, a more intelligible and relevant program of school activities, a more effective use of the child's natural modes of learning, and so on—is basically the same as the need for improved education for *all* children. But if this is not a separate problem, it does represent a separate and recognizable source of pressure on the schools. (Cremin's volume is relevant here, too—a concern for this kind of humane social problem was one of the major strands running through progressive education.)

The Revolt Against Formalism and Ineffectuality

The mathematics classroom of recent years has been one of the most culturally-deprived environments inhabited by any American child; it has offered little beyond blackboard, chalk, pen, paper and textbook. What is worse, the plot has been as barren as the scenery: in one high school of good reputation a teacher recently spent two consecutive 45 minute periods writing examples like this

$$x^2 \cdot x^5 =$$

on the blackboard and going around the room, letting each child in turn give an answer to a question of this type:

$$x^2 \cdot x^5 = x^7$$
$$p^2 \cdot p^{10} = p^{12}$$
$$x^3 \cdot x^2 = x^5$$
$$x^2 \cdot x^6 = _____$$

and so on. She construed that she was "teaching exponents," but her methodology was straight out of Pavlov, and may possibly be the proper way to teach algebra to dogs. Human children "conditioned" this way learn so well that they come to college, see

$$x^2 + x^5 = _____,$$

[10] For a further discussion of computers, both in today's technology and in tomorrow's schools, cf. *Scientific American*, September 1966 (the whole issue is devoted to this topic), and *Saturday Review*, July 23, 1966 (much of the issue deals with these questions).

and respond (quite incorrectly) by saying "x^7," as many a college mathematics teacher can testify.

The kind of school we shall have if we move into the era of computerized instruction with this kind of primitive pedagogical philosophy staggers the imagination. At least the Pavlovian teacher of exponents was an attractive and well-dressed young lady, and the boys and girls in her class doubtless found things to think about while they sat and waited their turn to recite, although most of these things had surely but little bearing on mathematics.

To understand the educational inadequacy I am trying to describe, one need only compare a child at play with that same child sitting in class during a mathematics lesson. The problem is not new, and it has not been solved. On the contrary, this is a "problem of 1967" that was much in evidence in the nineteenth century, and can be understood better if it is viewed in historical perspective. Writes Cremin:[11]

> In 1873, the school board of Quincy, Massachusetts, sensing that all was not right with the system, decided to conduct the annual school examinations in person. The results were disastrous. While the youngsters knew their rules of grammar thoroughly, they could not write an ordinary English letter. While they could read with facility from their textbooks, they were utterly confused by similar material from unfamiliar sources. And while they spelled speedily through the required word lists, the orthography of their letters was atrocious. The board left determined to make some changes, and after a canvass of likely candidates, elected [Francis W.] Parker to the Quincy superintendency of schools.
>
> Things soon began to happen. The set curriculum was abandoned, and with it the speller, the reader, the grammar, and the copybook. Children were started on simple words and sentences, rather than the alphabet learned by rote. In place of time-honored texts, magazines, newspapers, and materials devised by the teachers themselves were introduced into the classroom. Arithmetic was approached inductively, through objects rather than rules, while geography began with a series of trips over the local countryside. Drawing was added to encourage manual dexterity and individual expression. The emphasis through-

[11] Cremin, *op. cit.*, pp. 129-30. The program described here is almost unbelievably "modern." "Approaching arithmetic inductively" is a major feature of many "modern" mathematics programs (for example, David Page's), and is discussed in Bruner's well-known *The Process of Education* (Harvard University Press, 1963). The idea of having children begin with the study of sentences and words, while not clearly spelled out here, may have been consistent with the modern procedures used in the carefully-researched reading program developed by Harry Levin of Cornell University, as part of "Project Literacy." Introducing mathematics through physical objects is a key feature of various "modern mathematics" projects (but not all of them), including the ESS program of ESI, and the Nuffield Mathematics Project in England. Whatever Parker's "research" may have been, his intuitions about teaching and learning appear to stand at the forefront of practice even in 1967.

out was on observing, describing, and understanding, and only when these abilities had begun to manifest themselves—among the faculty as well as the students—were more conventional studies introduced.

The ultimate comment on this problem was probably made by Colonel Parker himself, when he wrote (in his report of 1879),

> . . . I am simply trying to apply well established principles of teaching, principles derived directly from the laws of the mind. The methods springing from them are found in the development of every child. *They are used everywhere except in school.*[12]

Parker later became principal of the Cook County Normal School in Chicago, and developed his approach still more fully in the practice school there. The program at Chicago is described by Cremin as follows:

> There are innumerable accounts of what went on in the practice school, most of them by enthusiastic disciples who tend to wax eloquent about programs and outcomes. Parker himself maintained that his effort was twofold: to move the child to the center of the educative process and to interrelate the several subjects of the curriculum in such a way as to enhance their meaning for the child.
>
> . . . The large assembly hall became the common meeting ground of children and adults alike. Its exercises were conducted with the utmost informality, the emphasis being on sharing and self expression. . . .
>
> From the morning assembly the youngsters passed to their classrooms, where the same techniques of informality prevailed. For reading and writing, the children created their own stories, and these, in the form of "Reading Leaflets" printed at the school, quickly replaced primers and textbooks. Spelling, reading, penmanship, and grammar were all thus combined as elements of communication, to be studied within the context of actual conversation and writing. Drill was recognized as a necessity, but always in the context of more immediate student interests.
>
> At a time when drawing was first appearing on the American pedagogical scene, Parker made art a central enterprise of the practice school, arguing that modeling, painting, and drawing were modes of expression, "three great steps in the evolution of man." Science was begun in the form of nature study, and under the brilliant leadership of Wilbur Jackman, the children conducted trips through neighboring fields and along the lakeshore. They made observations, drawings, and descriptions, thus correlating their work in science with their studies in language and art. They later carried certain of their investigations into the classroom, thereby beginning elementary laboratory work in physics and biology.
>
> Mathematics was frequently introduced in connection with this laboratory work, as well as with the occupations of the manual-training rooms. There youngsters actually made the equipment they needed for their studies in science,

[12] Quoted in Cremin, *op. cit.*, p. 130. (The italics were added by me. R.B.D.)

nature study, the drama, along with the ubiquitous bookends and samplers. Geography, too, began with firsthand knowledge of the surrounding countryside, and insofar as geography was conceived as a study of the world as the home of man, elementary economics and history were likewise introduced. So it was also with music, the drama, hygiene, and physical education: all were seen as vehicles for child expression: all began with what had meaning to the children themselves. The job of the teachers was to start where the children were and subtly lead them, through language and pictures, into the several fields of knowledge, extending meaning and sensitivities all along the way. It was an exciting experience to teach at the school, as testified by many of the faculty who served with Parker. There was an enthusiasm about the work that quickly passed to newcomers, and to the children themselves. Innumerable visitors came from far and wide and also caught the thrill of what was going on.[13]

I have argued that the problems which confronted Colonel Parker confront most of us today. They are not solved. John Hersey's account of the child filing index cards for the dentist,[14] or Bel Kaufman's account of English classes and school atmosphere in a contemporary urban school,[15] are both instances of accurate descriptive writing by well-informed and acute observers. The Second Coming of Colonel Parker has not occurred—but the pressures are there that say that it must.[16]

If the reader feels that the advances of Parker and others have been achieved and are now commonplace in United States schools, there are many ways he can obtain evidence to the contrary, either by visiting schools or by studying contemporary reports. Let me quote briefly from one particularly articulate and recent report: the article entitled "You Force Kids to Rebel," which appeared in *The Saturday Evening Post*.[17] The author, Steven Kelman, is now an undergraduate at Harvard, and is writing about his own recent education in a highly-regarded school system on Long Island:

> On both high school and college campuses, the official statements about almost any subject are so widely distrusted nowadays that citing them is the best way to have yourself marked as a dupe or a simpleton. Adults might

[13] Cremin, *op. cit.*, pp. 131-33.

[14] John Hersey. *The Child Buyer*. New York: Alfred A. Knopf, Inc., 1960.

[15] Bel Kaufman. *Up the Down Staircase*. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1964.

[16] Cf. also Cremin, *op. cit.*, pp. 218-19, concerning William Heard Kilpatrick's writings on the subject of what constitutes proper education in a rapidly changing society; also p. 259, concerning George S. Counts' address to the 1932 convention of the Progressive Education Association, for his list of "notable achievements" of progressive education.

[17] Reprinted with permission of *The Saturday Evening Post* (November 19, 1966), pp. 12ff. © 1966 The Curtis Publishing Co.

understand how serious this problem is if they'd listen to the words of the songs of somebody like Bob Dylan. His most popular songs are talking about skepticism, about what's really going on in the world as compared to what we're being taught is going on in the world. When we take a look for ourselves, the facts we see are so different from what we've been taught that we have no choice but to turn into rebels or at least skeptics. . . .

Many of us come to realize just how unreal the classroom world is when our thoughts turn to boy-girl relationships. No teen-ager can escape knowing that love and sex are part of the real world. So how does society's agent, the school, present this part of reality? It ignores it. For instance, one biology teacher I heard about treated his students to the obscene spectacle of his own sniggering while he described sexual reproduction in algae. Health teachers reduce puberty to a section of an inane chart on "stages of human development." When we find out the facts and feel the emotions, how can anyone expect us not to be skeptical about an adult world which tries to act as if none of this existed? And the moral code that we have developed, "sex with love," seems to us to be more logical than anything you've put up.

The whole idea the school seems to try to get across is that if you don't teach it to us, it doesn't exist. This can sometimes go to extreme length. In junior high school we had a thing called a "Reading Record Card." This was supposed to be a list (and brief discussion) of all the books you had read each year. But "all the books" actually meant all the books that were in the school library. And when students protested against the refusal to allow listing of books like *1984* and *The Grapes of Wrath*, we were treated like people in China who try to whisper that Mao Tse-tung is not the only recognized writer in the world. And what are we taught about literature? We are often required to memorize such details as "What color was Ivanhoe's horse?" and "What hotel did Gatsby and the Buchanans meet in?" rather than talking about how a book means something in helping us to figure out ourselves or other people. So kids often give up the classics. One kid told me that he feared becoming a writer because of what high school English teachers would do to his books.[18]

The Second Coming of Colonel Parker has not occurred. We have not built into United States schools the kind of program that Parker, or Dewey, or Kilpatrick was striving for. Perhaps we did come close, once or twice: in Parker's own school, in Dewey's school, in a few other places, and especially in the exceedingly important Eight-Year Study undertaken by the Commission on the Relation of School and College of the Progressive Education Association.

The Commission—and its celebrated Eight-Year Study—grew out of a discussion at the 1930 convention on how the high school might improve its services to American youth, or, put more bluntly, on how progressive education might be extended more effectively to the secondary level. We are told

[18] *Ibid.*

that suggestions were plentiful, but that apparently one difficulty lurked in the minds of all: the problem of college entrance requirements.[19] Therefore, it was suggested that the Executive Board appoint a committee "to explore the possibilities of better coordination of school and college work and to seek an agreement which would provide freedom for secondary schools to attempt fundamental reconstruction."

. . . The group set to work immediately, and after a year of study came up with a report sharply indicting American high schools on a number of familiar counts: they had failed to convey a sincere appreciation of the American heritage; they did not prepare adequately for citizenship; they seldom challenged gifted students to the limit of their abilities; they neither guided nor motivated their pupils effectively; and their curricula were a hodge-podge of lifeless material unrelated to the real concerns of young people.

Based on its analysis, the Commission proposed an experiment in which some twenty leading secondary schools, public and private, would be invited to redesign their offerings with a view of achieving (1) greater mastery in learning, (2) more continuity of learning, (3) the release of the creative energies of students, (4) a clearer understanding of the problems of contemporary civilization, (5) better individual guidance of students, and (6) better teaching materials and more effective teaching. "We wish to work toward a type of secondary education which will be flexible, responsive to changing needs, and clearly based upon an understanding of young people as well as an understanding of the qualities needed in adult life," the Commission declared. "We are trying to develop students who regard education as an enduring quest for meanings rather than credit accumulation; who desire to investigate, to follow the leadings of a subject, to explore new fields of thought; knowing how to budget time, to read well, to use sources of knowledge effectively and who are experienced in fulfilling obligations which come with membership in the school or college community."[20]

The experiment ended in 1940, and its results were published in 1942. The timing was unfortunate, for the whole affair was eclipsed—one might say wiped out—by World War II.

The actual results, in fact, were impressively positive. A team of measurement experts, led by Ralph W. Tyler, compared the graduates of the participating "progressive" schools with other college students of similar background and ability.

The team's technique was to set up 1,475 pairs of college students, each consisting of a graduate of one of the thirty schools and a graduate of some other secondary school matched as closely as possible with respect to sex,

[19] Doesn't this sound like the "mathematics revolution" of the 1960's? The noteworthy fact, though, is that the Eight-Year Study made *more* progress in attacking the problem, attacked it on a more fundamental level, and won more freedom from the tyranny of tests, than any of the "modern" projects have done.

[20] Cremin, *op. cit.*, pp. 251-53.

age, race, scholastic aptitude scores, home and community background, and vocational and avocational interests.

In comparing the 1,475 matched pairs, the evaluation team found that graduates of the thirty ["progressive"] schools (1) earned a slightly higher total grade average; (2) received slightly more academic honors in each of the four years; (3) seemed to possess a higher degree of intellectual curiosity and drive; (4) seemed to be more precise, systematic, and objective in their thinking; (5) seemed to have developed clearer ideas concerning the meaning of education; (6) more often demonstrated a high degree of resourcefulness in meeting new situations; (7) had about the same problems of adjustment as the comparison group but approached their solution with greater effectiveness; (8) participated more and more frequently in organized student groups; (9) earned a higher percentage of nonacademic honors; (10) had a somewhat better orientation toward choice of vocation; and (11) demonstrated a more active concern with national and world affairs. Moreover, the graduates of the *more* experimental of the thirty schools showed even greater differences [e.g., greater gains] along these lines from the students with whom they were matched.

In a summary report to the Association of American Colleges early in 1940, Dean Herbert E. Hawkes of Columbia College concluded: "The results of this Study seem to indicate that the pattern of preparatory school program which concentrates on a preparation for a fixed set of entrance requirements is not the only satisfactory means of fitting a boy or girl for making the most out of college experience. It looks as if the stimulus and the initiative which the less conventional approach to secondary school education affords sends on to college better human materials than we have obtained in the past." [21]

Anyone who believes that American education went ahead to build from this point onward is naive and ill-informed. To say that the Eight-Year Study was wiped out by World War II and a shift of national interest away from the schools is by no means to exaggerate. We have continued to operate in a pre-Eight-Year Study mode, and we do so today.

But the pressure for change is there. If we can calm down today's international scene a bit, there is reason to hope that we can again focus national attention on the matter of schools and the education of children.

If we can, we shall be back to the point of the end of the Eight-Year Study—except that we shall have the technology of the 1960's and 1970's to contend with, rather than the technology of 1940 or 1942.

The main lessons seem to be clear: schools appear to be afflicted with a tendency toward becoming academic, irrelevant, hypocritical, uninspiring and ineffective. They must continually devise ways to fight against this tendency. The "progressive education" movement was one such attempt to introduce into the schools modernity, relevance, vitality, vigor, and a sense

[21] Cremin, *op. cit.*, pp. 225-56.

of commitment. The main justification of the curriculum revision projects of the 1960's is that they represent another such attempt. The more one analyzes their methods and their difficulties in detail, the more closely they seem to resemble the efforts of Parker, Dewey, Kilpatrick, and the Eight-Year Study.

There is one major point of difference: the new biology, physics, mathematics (etc.) projects are confronted—*as are the schools themselves* —with the highly sophisticated technology and society of the late 1960's. It would no longer be "progressive" to win the battle to make the schools relevant to the society of 1942. Nothing less than 1967 is acceptable—and we should probably be thinking at least in terms of the 1970's, for the graduates of our schools are young, and need to be prepared for the world of the future.

One could ask: why do schools exhibit this persistent tendency to become old-fashioned, academic, unrelated to modern society, and lacking in vitality? This is quite clearly a most important question, but to pursue it would lead us too far afield from our present consideration of modern mathematics curricula. Unless, however, these basic questions are dealt with, SMSG, UICSM, UMMaP, and all the others will add up to some carefully worked-out chrome plating added to an educational vehicle that doesn't run very well.

Lest we become lost in pages of generalities, it may be well to jump in and look at a few actual projects and school programs.

Some Specific Projects and Some Specific School Programs

1. **Computer-assisted Learning.** We have already referred to the work in computer-assisted learning ("CAL") being done by a team of workers at the University of Illinois, Urbana, Illinois, and to other CAL work being done at Stanford University, under the direction of Patrick Suppes. CAL research is going on elsewhere, as well; notably at MIT, and at the Stony Brook campus of the State University of New York. Within industry, IBM is reportedly making great strides with CAL, and in fact is cooperating in the work at Stony Brook. This list is by no means complete.

2. **Studying the Younger Children.** Perhaps we can best begin with the younger children. A program of unusual interest operates in four nursery schools (with children ranging in age from about 2½ years old to about 5 years old) in greater St. Louis, under the direction of Donald Bushell of Webster College. The theory of child motivation employed is consciously and explicitly the "operant conditioning" of B. F. Skinner; a large part of the arithmetic learning is accomplished by means of a rather simple CAL arrangement, in the form of a machine that presents a large slide-projected picture to the child as a "problem" or "task." Three smaller slide-projected pictures are available as "responses" or "answers," and the child responds by actually pushing on one of these smaller pictures, which actuates the electronic machinery to advance the program. The "reinforcement schedule" or "reward schedule" is of interest, but we shall ignore it here, in order to move on to a more specifically mathematical question: *How does a child build his earliest notions of number?* There are several different theories, including at least these six: the child builds his earliest ideas of number

a. By *abstraction,* from experience with two pencils, two dogs, two boys, two apples, and so on, by asking "what do these have in common?";[22]

[22] Yet it is worth noting that considerable evidence suggests that what is most commonly observed in many situations is not "What is the same?" but rather the discrepant event, "What is different?" or "What is unusual?"

b. By *experience in performing the act of counting*, which is here regarded as a human act that is learned by imitation, much as a child learns to swing a baseball bat, or to hold a spoon; *after* the act has been performed, it can be discussed, much as a class might subsequently discuss a trip to the zoo, etc.;

c. By getting first the idea of "more," "less," and "equality," and *thereafter* giving number names to "as many as I have fingers," etc.;

d. By studying sets, and the various attributes of sets, including the numerousness of the things in the collection (other properties of sets are studied in biological classification schemes, etc.; "numerousness" is by no means the *only* property of sets);

e. By studying *invariance*, as in the fact that rearranging pebbles in a different *pattern* or *order* does not change *the number of pebbles present;*

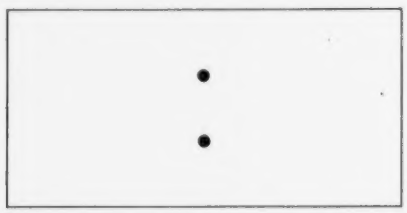
f. By a *gradual deformation of perceptual stimuli.*

This last interpretation is perhaps the most novel. It has been used by the Madison Project in forming concepts of place-value numerals by using the Dienes Multi-Base Arithmetic Blocks ("MAB" blocks),[23] but a much more striking use of it has been made by Professor Bushell and his colleague, Miss Jo Maiorano.

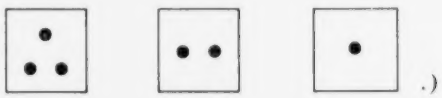
In considering the Bushell-Maiorano program, it is worth recalling that CAL equipment records and analyzes *every student response*. This, from the point of view of an author or researcher, is perhaps the greatest value of CAL equipment. Whereas the best textbooks ordinarily contain gaps, redundancies, or murky places, CAL equipment enables an author to pinpoint with unprecedented accuracy the *exact* spots in a program where students encounter difficulty. Not only are right and wrong answers recorded, but response time is, also. A question that requires protracted thought is not necessarily a bad question, but all questions with large response times are identified immediately. In the Bushell apparatus, the *next* question does not appear until the student requests it (by pushing against the large screen); and the delay in requesting the next question is also recorded and analyzed.

Bushell and Maiorano found that, if the large-screen "task" was a pattern of two dots, one below the other in a vertical line,

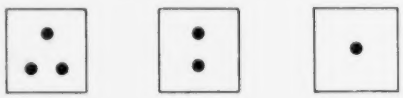
[23] Dienes MAB blocks are available from: Education Nouvelle, 306 Est, Rue Sherbrooke, Montreal 18, P.Q., Canada. You may also wish to consult the catalogue of the Educational Supply Association, Ltd., School Materials Division, Pinnacles, Harlow, Essex, England. Professor Dienes' address is: Professor Zoltan P. Dienes, Faculte des Sciences, Universite de Sherbrooke, Sherbrooke, P.Q., Canada.



then children did not readily match this against a horizontally-arrayed two-dot "answer" on one of the three small screens (as in the arrangement



However, if the three small screens displayed something like this

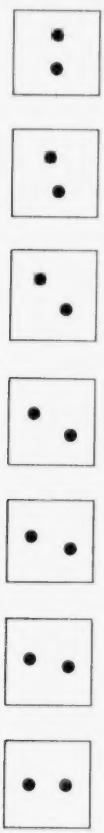


the children did not hesitate in pushing on the middle screen. From this background, Bushell and Maiorano developed a sequence of "questions" where the actual perceived stimulus varies very slightly. The same vertically-arrayed large-screen "question" might be matched up, first, with an identical vertically-arrayed "answer." Then, later, while the large-screen "question" remains in the same vertical array, the small-screen "answer" appears as

then later as

and finally as

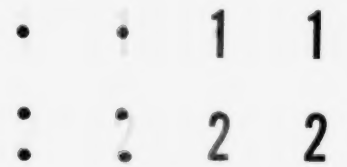
The Bushell apparatus uses hand-made slides. With larger computer capacity, it would be possible to use instead electronically-generated dots, and the computer could experiment with sequences of very closely similar stimuli



or with more rapidly-moving sequences where fewer intermediate steps were interpolated between the vertical and the horizontal arrays. Moreover, as data accumulated, an electronically-generated sequence could match the number of interpolated steps *against the recorded past history of each individual child.* This is really individualizing instruction!

The Bushell-Maiorano program introduces numerals by superimposing a light (and barely discernible) numeral over a heavy (and clearly visible) pattern of dots:

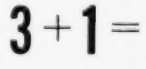
and so on. Now the program again goes through a continuous deformation sequence: on subsequent occurrences, the dots become lighter and gradually recede to the level of imperceptibility, while the numerals become firmer and more distinct:



Addition facts are introduced in the same way: a "problem" that will appear first as

will need to be matched against an answer

which is almost nothing more than perception. Again, the same sequence of gradual stimulus modification is made, with the numerals gradually appearing, the "+" and "=" signs (which were at first almost imperceptible) gradually becoming firmer and more prominent, and the dots gradually receding into invisibility, until the final "question" appears as



and the "answer" appears as

This approach to stimulus- or task-sequencing might be called "$\in$-connected stimulus sequencing," to adapt a name from mathematical topology.[24]

I have gone into the Bushell-Maiorano experimentation in some detail for several reasons: no published reports of this work exist as yet, there is currently a great interest in nursery school programs (stimulated by Project Head Start), the learning principle involved is an unusual one (although it has been used before), and it seems important to recognize that there is wide disagreement concerning basic learning paradigms, *even on so fundamental a matter as the child's first conceptualization of number!* It is by no means true that "learning theory tells us how to construct a scientifically-designed curriculum."[25]

There is yet one more reason for considering the Bushell-Maiorano program. It forces us to broaden our view of "new mathematics programs," inasmuch as the Bushell-Maiorano program *rejects sets* and builds instead upon perceptual discrimination ability, and the experience that can be derived therefrom.

We shall return later to the question of the value of "sets" in the primary grades. "Sets" have become, in the popular mind, one of the characteristic stigmata of "the new mathematics." *This represents a narrow view of what is in fact occurring.* Indeed, until we learn more about how children normally do form concepts of number, it would seem that we are committing the common educational offence of superimposing an externally-determined behavior on a living being, without considering the meaning and relevance that this behavior has for that being. Many educators will—quite correctly—find it hard to accept an externally-imposed curriculum that is not wedded to sensitive and perceptive study of children's normal modes of behavior.

3. **The Ideas of Jean Piaget.** Partly due to the work of Z. P. Dienes, Leonard Sealey, and others, the ideas of Jean Piaget have had considerable

[24] Notice that this is quite different from the sequencing of verbally-coded *questions* in close sequence, so that the step from one question to the next is very small. In the Bushell case, it is *actual perceived stimuli* that are changed slightly, probably taking advantage of some inaccuracy or uncertainty or acceptability range in *perception;* the small-step sequencing of *questions* is evidently a quite different matter. Robert Gagné did some well-known question-sequencing work with John Mayor's University of Maryland Mathematics Project ("UMMaP"). For an excellent discussion of perceptual categories in the case of discrimination between phonemes, cf. Roger Brown, *Social Psychology* (Free Press, 1965), pp. 260 ff.

[25] It is interesting to observe children using the Bushell-Maiorano program. Many children count aloud, spontaneously. Frequently they count incorrectly, as small children often do. Nonetheless, they *do* select the correct answer in "dot-pattern" form. Apparently some sort of "dot-pattern" visualization appears *before* the child gains the accurate movement synchronization necessary for correct counting.

influence in the actual day-to-day operation of many English schools. The contrast between English and United States schools is very striking, and should form the basis for some valuable cross-cultural studies. One matter worth noting—and worth explaining, if possible—is the fact that both curriculum content and pedagogical methodology in English schools have changed gradually (but very substantially) over the past decade or two, while both curriculum content and methodology in the United States have changed very little. Without pretending to "explain" this in the present short discussion, we might note that English schools tend to be smaller than United States schools; that English headmasters and teachers enjoy more autonomy than most of their United States counterparts; that many English headmasters (and quite a few teachers) are genuine scholars, well-informed about developments in Denmark, Switzerland, Russia, Germany, and the United States; and that the large, business-like United States school so well described by Raymond Callahan[26] has hardly appeared as yet in England (although it now seems to be making its debut).

It would be interesting to see some well-handled studies of the economics of textbook publishing in England, vs. the economics of textbook publishing in the United States. There is considerable economic significance in the English use of the "integrated school day"—whereby a few children are studying art, while others in the same classroom are studying reading, others are playing with blocks, and others are working on mathematics. This means (among other things) that an English classroom of 40 children needs only perhaps 8 arithmetic books, one desk calculator, 8 reading books, etc. It is not necessary to purchase a "classroom set" of 40 books, since all 40 children never do the same thing at the same time. There is also curricular significance in another English innovation: the use of "family plan" school arrangements, whereby a single room will contain children of widely-varying ages—just as a normal family does.[27] The older children have some responsibility to help with the younger children.[28]

This, however, is only background to our present concern. One of the finest and most carefully devised of all of the "modern mathematics" projects is, unfortunately, one of the least well-known. This is the English project known as the Nuffield Mathematics Teaching Project (Lord

[26] Raymond Callahan. *Education and the Cult of Efficiency*. Chicago: University of Chicago Press, 1962.

[27] . . . and just as United States schools did in the eighteenth century. Cf. John I. Goodlad and Robert H. Anderson. *The Nongraded Elementary School*, revised edition. New York: Harcourt, Brace & World, Inc., 1963. p. 44.

[28] Incidentally, this "family plan" arrangement is used also by Donald Bushell in his nursery schools, described earlier.

Nuffield, of Morris Garages, being roughly the English equivalent of Henry Ford), directed by Geoffrey Matthews of St. Dunstan's College, in London. Many observers consider this the best of all primary-grade mathematics projects. Certainly it is the most closely wedded to careful child study. It avoids the common United States practice of "jumping in and trying to change the child" before you have any understanding of the child you are dealing with.

It may be best to allow the Nuffield Mathematics Project to speak for itself:

> The object of the Nuffield Mathematics Teaching Project is to produce a contemporary course for children from five to thirteen. This is being designed to help them connect together many aspects of the world around them, to introduce them gradually to the processes of abstract thinking, and to foster in them a critical, logical, but also creative, turn of mind.
>
> A synthesis is being made of what is worth preserving in the traditional work with various new ideas, some of which are already being tried out. These cover presentation as well as content, and emphasis will be placed on the learning process. A concrete approach will be made to abstract concepts, and the children should be allowed to make their *own* discoveries whenever possible. The work of the project is set against the present background of new thinking concerning mathematics itself. . . .
>
> The work of the team members of the project is centered round the production of teachers' guides. . . .
>
> *The main requirement for the reader interested in the work of the project is an open mind.* Primary "arithmetic" was fossilized for so many years that the present opportunity to rethink what mathematics is really appropriate for children is an overwhelming one: the first requirement is for genuine understanding on the part of the children.
>
> For example it is easy to drill children into writing a statement like "3 + 2 = 5" before they can really appreciate the meaning of the symbols. Does this mean "Take 3, add 2 and you get 5" or "3 plus 2 gives 5" (what is "plus"?) or "3 added to 2 makes 5" or "whenever I have 3 things and then get 2 more, I end up with 5 things" or none of these things?
>
> And is it right to encourage children to write such statements before they can recognize the "fiveness" of five and identify the existence of five objects?
>
> Symbols should not be introduced before they are really meaningful and unambiguous. There are plenty of mathematical experiences which should come before "3 + 2 = 5" and which are perhaps more fundamental. By starting apparently more slowly, quicker progress based on understanding will follow later. This understanding will grow through realization of the power and scope of mathematics. Mathematics is *more* than "doing sums." In fact perhaps the most important message of "modern" mathematics is its ubiquity.
>
> The word "set" has become an emotive one, and indeed some authors

seem to have used set theory to make simple ideas as incomprehensible as possible. . . .

If "mathematics" really embraces more than we formerly allowed, it may be that some of the newer ideas are actually easier for children to grasp than the traditional torrent of "number," and this is our hope.[29]

Some special features of the Nuffield Project deserve special stress:

a. Franklin Morley, of the Ladue, Missouri Public Schools, has suggested that one way to classify "new mathematics" projects is *by the point at which they seek to intervene in the educational operation.* Under this scheme of analysis, SMSG has, for the most part, sought to intervene at the point where an author sits down to write a text—that is to say, the main focus of SMSG has been on the production of "sample texts" by which they hoped to influence authors and publishers (and, obviously, textbook selection committees).[30]

Other projects have sought to intervene at other points. The Nuffield Project seeks to intervene at the point of instructional planning by the teacher (or by the headmaster). As indicated by the excerpt quoted above, the Nuffield Project *has not focused on producing texts for children.* Rather, this project has focused on preparing a set of books designed to help the teacher in his lesson planning. In addition, its staff members have taken rooms in certain school buildings, scattered throughout England, and have equipped these rooms with the kinds of materials necessary for building the kind of school experiences they advocate. These rooms also serve as planning centers for teachers and headmasters.

Going still further, they have prepared a short fourteen-minute film entitled *I Do—And I Understand.*[31]

This film shows an actual classroom lesson, of the type the Nuffield Project advocates. The class is broken down into small groups of 3 or 4 children each. Each group selects an "assignment card" that describes the task they are to work on. All tasks involve the use of physical materials. In overall appearance, the classroom scene resembles

[29] Quoted from *Beginnings* (1), one of the first booklets produced by the Nuffield Project, page 1. This booklet is considered merely a "first draft—not for publication." It is dated 1965. Despite its labeling as a "first draft," it, and the other pamphlets in this series, deserve very careful study. They are available from: The Nuffield Foundation, Mathematics Teaching Project, 12 Upper Belgrave Street, London, S.W.1, England.

[30] Cf. *Newsletter No. 24: October 1966,* School Mathematics Study Group (SMSG), School of Education, Cedar Hall, Stanford University, Stanford, California.

[31] The title is taken from the Chinese proverb: "I hear, and I forget; I see, and I remember; I do, and I understand." The film is available from: Sound Services, Ltd., Wilton Crescent, Merton Park, London, S.W.19, England.

visually either chemistry laboratories or woodworking shops, if one seeks a near-parallel among common United States classroom scenes. Children are standing more often than sitting. Many children move around the room. Each small group plans and executes its own project, as directed by its assignment card. The teacher circulates around and talks with one group at a time, then passes on to talk with another group. The film moves too quickly to enable the viewer to identify completely each task—such was not its purpose—but we might identify a few of the tasks as something like the following:

One group of children is measuring the diameters of various balls with a gigantic pair of calipers, measuring the circumferences using string, and comparing.

Another group of children (working outdoors) appears to be trying to determine how many bricks were used in constructing the entire school building.

Another group of children is measuring distances between cities on a map, using a small wheeled tracking machine to follow along actual highway routes.

Another group of children is using a stop-watch to measure the speed of an electric train as it races around an oval-shaped track layout.

Yet another group of children—also working outdoors—is measuring an angle of elevation, and then using similar triangles and ratio and proportion to calculate the height of the school flagpole (or some such object).

Other groups are: working with a balance beam; pouring water into jars in studying volume; pushing the ubiquitous English "trundle wheel" (which is one yard in circumference) in order to measure distances; and working on some problems involving area.

A pamphlet is available that describes this film and the method of setting up a similar classroom situation in your own school. The pamphlet bears the same title as the film: *I Do—And I Understand.*[32]

b. The Nuffield Project, as indicated above, bases its program upon experience, working with physical materials.[32a]

c. The Nuffield Project has children work with one-to-one correspondence, and partitions of sets, *before* it turns to a discussion of sets themselves.

[32]The pamphlet is available from: The Nuffield Foundation, Mathematics Teaching Project, 12 Upper Belgrave Street, London, S.W.1, England.

[32a]Among physical materials recently introduced in the United States, cf. for example: H. A. Thompson, L. Foster and S. Pollock. *Color-Factor Mathematics.* New York: McGraw-Hill Book Company, Inc., 1964; Orlie W. Laing, Ralph C. Williams and Frank A. Yett. *Mathematics Kit.* Boston: Allyn and Bacon, Inc., 1966.

This last matter deserves discussion. Much thinking in the United States has used arguments such as: "The children are actually counting sets, whether they use the word 'set' or not," and "How can you talk about one-to-one correspondence before you talk about sets?"

These arguments should have been rejected from the outset. They are based upon a confusion that fails to distinguish *physical interaction between the child and his environment* from the quite different matter of *abstract intellectual analysis of the child's acts.* The child, after all, would nowadays be analyzed as "consisting of cells." Does this mean he should begin with the study of cells, cell nuclei, osmotic pressure, glucose, adenine, guanine, and DNA? The child makes sentences. Does this mean he should begin at age 3 to study grammar and linguistics? The young child crawls—does this mean he should begin with the vector calculus description of the motion of material objects?

Obviously, *what a child is or does* is an entirely separate matter from *how contemporary adults would go about constructing an abstract symbolic description of what the child is doing.*

Long before "talking about sets," the Nuffield Project has children use yarn to connect a drawing of each child to a drawing of a toy that that child owns. Children take a pile of physical objects and sort them out in various ways—for example, into a heap of plastic things, another heap of wooden things, and a third heap of metal things.

Speaking for myself, I do not know what is the best way to approach a child's introduction to the concept of *number.* What is important, though, is that there are many possibilities: you can use paper-and-pencil tasks, or you can use tasks with physical objects. You can discuss what you are doing by means of sophisticated language, or you can avoid the use of sophisticated language. You can build on perception, as Bushell does; or on sets, as many United States texts now do; or on the process of *counting*; or on the use of *physical* "one-to-one correspondences" *without using sophisticated descriptive language*—which is the Nuffield Project approach.

What cannot help but strike the outside observer is the premature rush to "get sets in somehow" that seems to have characterized so much primary-grade work in the United States. *Few people even seem to have tried to list the possible alternatives!* Moreover, the argument that "the children really *are* working with sets, no matter what we call them" is dead wrong: a child playing with buttons is engaged in a physical action that can be analyzed descriptively in many different ways. (The child might not even know those "things" were "buttons," and he surely does not necessarily think of "the aggregate of them" as constituting a "set.") Of course we shall gradually help the child to

use abstract descriptions of what he is doing—but *which* descriptions should come *first*? Separating the "buttons" into "leather buttons," "bone buttons," "metal buttons," and "plastic buttons"? Thinking about the different *colors* of the buttons? Thinking about "big" buttons vs. "little" buttons? Learning to manipulate buttons in putting on and taking off our clothes? Collecting and trading buttons and developing some economic understanding of currency?

To say that the child "really *is* working with sets," is to betray a misunderstanding of the nature of human knowledge, and of the process by which man fashions abstract symbols of various sorts as a pale reproduction of the richness of reality. There are many possible descriptions of what the child "really *is* doing." That one particular description has a (probably temporary) ascendency in the analytic researches of contemporary logicians tells us extremely little about the nature of human children, and the ways in which they explore, learn and grow.[33]

d. The Nuffield Mathematics Project offers at least one additional point of interest: it uses a method for organizing the school curriculum which is largely original and very suggestive.

The problem of *how to organize the many diverse bits and pieces of the mathematics curriculum* will become a severe one, indeed, if a real "mathematics revolution" ever does get started. At present, the curriculum in the United States is arranged mainly by grade-level placement of topics. The geometry studied most extensively in the United States (though virtually nowhere else in the world) is synthetic Euclidean geometry, it is studied by an axiomatic method, and it is studied largely in grade 10—indeed, it commonly occupies all of grade 10 mathematics. There are, however, many alternatives: instead of Euclidean synthetic geometry, one might select for study Cartesian analytic geometry, or vector geometry, or projective geometry, or affine geometry, or "motion geometry," and one may use many different approaches: the "axiom-proof-theorem" approach, the "empirical veri-

[33] Moreover, before we bet all of our chips on the "sets" description, we may want to think about the remarks of such eminent mathematicians as Saunders MacLane, who has suggested that perhaps sets do *not* form the best possible foundation for the analytical dissection of mathematics, and that perhaps alternatives (such as "categories," which are quite different) need to be considered. Warwick Sawyer and others have suggested that we consider essentially a "sociological" or "decision procedure" analysis, whereby we study *what mathematicians actually do*, rather than the shadowy abstractions which they have in mind. The growing use of digital computers may constitute a force that will move us in the direction Sawyer suggests, since computers carry out decision procedures but *don't have any concepts in mind at all*!

fication" approach, a "problem-solving" approach, a "descriptive" approach, and so on.

Moreover, we can put all of the geometry into grade 10 (as is more-or-less true at present), or we can spread it out over many years. We can integrate the study of geometry so completely into our study of arithmetic and algebra that no dividing line will be visible and arithmetic-algebra-geometry will merge into a single unified subject, or we can keep geometry separate and clearly distinct from the rest of our mathematics.

The present grade-level placement of topics in the United States provides a purely conventional answer to the question of what should go where. There is no foundation to this sequence other than simple accident.

Describing this method of answering "what should go where," Goodlad and Anderson wrote:

> When the Quincy Grammar School[34] opened its doors to pupils in 1848, certain enthusiastic citizens predicted that its new organization would set the pattern for fifty years to come. More than one hundred years later, the basic pattern is scarcely changed! The Quincy school was graded. . . .
>
> The Quincy Grammar School did not "just happen." Movements toward grading were clearly in evidence during the preceding century. In the eighteenth century, the selectmen of Boston developed separate reading and writing schools. Boys attended one and girls the other, changing at midday. New buildings provided reading schools on the upper floor and writing schools on the lower.
>
> A certain ordering of instruction began to appear: arithmetic was to be learned at the age of eleven; ten lines were to be written from copybooks in a single session, and ciphering done every other day. Certain accomplishments were deemed appropriate for specific levels, and the emphasis was on subject matter and skills. In fact, grade "norms" were being introduced. . . .
>
> Still another development, already mentioned, had its own considerable influence on the movement toward graded structure. This was the appearance of new textbooks. Speller, reader, grammar, and geography texts made their appearance. . . . The remarkably modern arithmetic texts of Warren Colburn made their appearance in 1821. Then, in 1836, the first works that were to become *The McGuffey Eclectic Readers*, graded through six levels and glamorized with abundant illustrations, began their fifty-year domination of juvenile (and adult) literary life. The phenomenal sales of these early works inspired others to produce them and the textbooks poured in upon the schoolmaster. Persuasive salesmen and uninformed teachers

[34] Notice that we are now talking about United States education *before* the appearance of Colonel Parker.

together compounded a situation of complete confusion that only gradually cleared when uniform textbooks ultimately were selected from recommended lists.

Textbook series—first in reading and arithmetic and later in science, social studies, health, and so on—came to be rigorously ordered by grades. *The work considered appropriate for a given grade level determined the content of the textbook, and then the content of the textbook came to be regarded as appropriate for the grade. In time, more fundamental procedures for determining the curriculum were scarcely considered.* [Italics mine. R.B.D.] Teachers and parents alike came to equate adequacy of pupil performance with ability to use the book designated for the child's grade level.[35]

To make matters worse, this arbitrary arrangement has gradually been built into the procedures by which we obtain norms for widely-used standardized tests.

How rational is present grade-level placement? Knowing how the arrangement was made, we should not expect too much internal logic—and we shall not find much. The difficult concept of dividing one fraction by another appears around grade five, whereas it surely should appear at a later date (which calls up the ominous specter of those standardized tests). The far easier topic of adding signed numbers is often delayed until grades 8 or 9. There is no foundation for all of this beyond accumulated historical accident.

Now, by contrast, the Nuffield Project attempts to make some use of Piaget's studies of child growth and development (and similar studies undertaken by other investigators) in determining the sequential order of the curriculum. Before asking how adequate a curriculum framework this provides, let us look at some of the child-centered developmental flavor of the Nuffield approach:

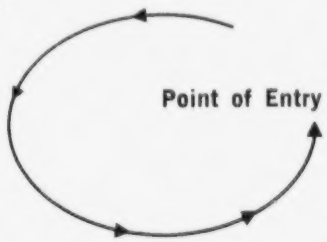
When children are doing creative work (disregarding all "laws" of proportion and perspective) it would be ludicrous to intervene on mathematical grounds.[36]

The key to awareness of space is movement. Opportunities are provided in school for young children to experience the joy of moving in as large a space as possible. As children enter the hall or playground for movement they use the available space in a particular way. They seem instinctively to move freely round so as to form a circle.[37] (See illustration on page 34.)

[35] John I. Goodlad and Robert H. Anderson. *The Nongraded Elementary School,* revised edition. New York: Harcourt, Brace & World, Inc., 1963. pp. 44-47.

[36] *Beginnings* V. Nuffield Mathematics Teaching Project. p. 10. (Note that this volume is different from the one entitled *Beginnings* (1).)

[37] *Ibid.*, p. 16. One might argue that what is involved here is more a matter of *momentum* than of *space*; or perhaps it has sources in man's evolutionary past.



This is a difficult task for children, and their [individual] success or failure will give clear indication to the teacher of the level of development of the [individual] child attempting it [remember that all instruction in Nuffield classrooms is on an individualized basis!].[38]

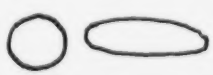
It might be helpful at this stage of development to determine whether a child has as yet established the notion of weight and its invariance. The following is a simple test deriving from the work of Piaget.

Ask the child to make two balls of plasticine which are "of the same weight" (but see below). If he has attempted this by using sight and muscle pull only, then ask him to put the balls of plasticine on the balance scales. If the balls do not balance ask him if he can make them balance.

At an early stage he will not be able to see that by taking a little from one ball he might make them balance. Do not proceed further with this child. He needs a great deal more play experience with materials.

At a later stage of development the child will be able to make the necessary adjustments to ensure that the two balls of plasticine make the scales balance.

Put the scales to one side and concentrate on the plasticine balls. Ask the child to roll one of the balls into a sausage. Then ask "Which is heavier, the ball or the sausage?"



The answers given by the child will give clear clues to his thinking. He might say that there was more plasticine in the sausage so it must be heavier. He might say that they still weighed the same, but there was now more plasticine in the sausage. There will be all kinds of variation in response.

[38] *Ibid.*, p. 57.

If the child is certain that the amount of plasticine has not changed, nothing has been added, nothing removed, and that as they were the same weight originally so they must be the same weight now, then he is at a mature level of thinking.

However, it might be profitable to proceed further on a subsequent occasion. Begin again with the two balls of plasticine. Determine and accept that they are "of the same weight." Then break one ball of plasticine up into smaller pieces.

Similar questions will provoke another wide variety of response and will indicate the child's level of thinking at the time. Whatever his response it is probable that when the small pieces of plasticine are rolled back into one ball the child will be sure that we have returned to our starting point.

It must be emphasized that this kind of understanding cannot be achieved through teaching. Children need years of varied experience with many kinds of materials before they can arrive, with certainty, at this notion of invariance.[39]

In short, the Nuffield Project seeks to identify clearly-defined developmental stages in the child's growth, and to hang its curricular plans on these pegs—individualizing for each single child separately, so that the children do *not* move together as a group.

This raises the question: how adequate are these "clearly-defined developmental stages" in providing us with curriculum guidelines? The answer seems to be that this method has great promise for the future, but that this promise has not been realized as yet. The presently-selected checkpoints do not seem to give us the guidance we need in devising a mathematics curriculum. There are many reasons for this inadequacy, including these:

(1) Most "new math" and "new science" projects have found

[39] *Ibid.*, pp. 58-59.

that student performance is at a high level in grade 5, *but actually diminishes as the child passes from grade 5 into grades 6, 7, and 8.* Many theoretical explanations of this phenomenon have been suggested, but the matter appears to be unsettled at the present time.

(2) There have been studies of the child's "ability to form concepts"—perhaps especially well-known are the studies by the Russians, Sakharov, Kotelova, Pashkovskaja, and Vygotsky[40]—and they have often reported results such as the following:

> The principal findings of our study may be summarized as follows: The development of the processes which eventually result in concept formation begins in earliest childhood, but the intellectual functions that in a specific combination form the psychological basis of the process of concept formation ripen, take shape, and develop only at puberty. Before that age, we find certain intellectual formations that perform functions similar to those of the genuine concepts to come. With regard to their composition, structure, and operation, these functional equivalents of concepts stand in the same relationship to true concepts as the embryo to the fully formed organism. To equate the two is to ignore the lengthy developmental process between the earliest and the final stage.
>
> Concept formation is the result of a complex activity in which all the basic intellectual functions take part. The process cannot, however, be reduced to association, attention, imagery, inference, or determining tendencies. They are all indispenable, but they are insufficient without the use of the sign, or word, as the means by which we direct our mental operations, control their course, and channel them toward the solution of the problem confronting us.[41]

This appears to contradict the experience of many "new mathematics" projects. For example, in the film *Second Lesson,*[42] a third grade girl named Ruth displays what has seemed to mathematician-observers to be an extremely mature and sophisticated mode of thinking about subtle and profound mathematical questions.

The entire issue is badly confused by the fact that what Vygotsky means by "a concept" seems to be an entirely different thing from what most mathematicians mean when they speak of "a mathematical concept." In the *mathematical* sense, a "concept" appears to mean what Suchman calls an "organizer," an abstract idea that enables one to deal more effectively with a large body of mathematical ex-

[40] Cf. Lev Semenovich Vygotsky. *Thought and Language.* Translated by Eugenia Hanfmann and Gertrude Vakar. Cambridge, Mass.: MIT Press, 1962, p. 58 and elsewhere.

[41] *Ibid.*, p. 58.

[42] Available as B & W, 16 mm sound motion picture film from The Madison Project, 8356 Big Bend Boulevard, Webster Groves, Missouri 63119.

perience. The *mathematical* "concept" must be natural, never arbitrary. (As Jerrold Zacharias has said, "Science is a game played against nature, never against the teacher.") Examples of mathematical concepts include: function, mapping, isomorphism, linearity, implication, rate of change, etc.

By contrast, Vygotsky's "concept" seems to be an arbitrary rule in the mind of the experimenter which the subject is supposed to guess.

While these two different meanings of "concept" may not be entirely dissimilar, they are by no means identical. The accumulation of a *large* amount of "readiness" experience which should precede mathematical concepts can have no parallel in the largely artificial world of Vygotsky's "concepts," and the vast difference in the *amount* of relevant experience is probably of decisive importance.[43]

(3) There is the further great difficulty that what anyone *says*, what he *knows*, and what he *does* are three distinct matters. The distinction is perhaps especially great in the case of children. We can easily find ourselves dealing with superficial questions of the child's *language* when what is clearly needed is a method for penetrating more deeply into his *thinking*. As Vygotsky describes it:

> Until recently the student of concept formation was handicapped by the lack of an experimental method that would allow him to observe the inner dynamics of the process.
>
> The traditional methods of studying concepts fall into two groups. Typical of the first group is the so-called method of definition, with its variations. It is used to investigate the already formed concepts of the child through the verbal definition of their contents. Two important drawbacks make this method inadequate for studying the process in depth. In the first place, it deals with the finished product of concept formation, overlooking the dynamics and the development of the process itself. Rather than tapping the child's thinking, it often elicits a mere reproduction of verbal knowledge, of ready-made definitions provided from without. It may be a test of the child's knowledge and experience, or of his linguistic development, rather than a study of an intellectual process in the true sense. In the second place, this method, concentrating on the

[43] The works of Vygotsky and his Russian colleagues may represent the earliest use of the "attribute blocks" which have since been used by Dienes, by ESI, and by others, and which are now beginning to appear in the primary grades as part of some "new math" programs. Somewhat similar *cards* are discussed in Jerome S. Bruner, Jacqueline Goodnow, and G. A. Austin, *A Study of Thinking* (New York: John Wiley & Sons, Inc., 1956). Since the value of attribute blocks in the mathematics curriculum is a matter currently under discussion, we shall return to this topic presently.

word, fails to take into account the perception and the mental elaboration of the sensory material that give birth to the concept. The sensory material and the word are both indispensable parts of concept formation. Studying the word separately puts the process on the purely verbal plane, which is uncharacteristic of child thinking. The relation of the concept to reality remains unexplored: the meaning of a given word is approached through another word, and whatever we discover through this operation is not so much a picture of the child's concepts as a record of the relationship in the child's mind between previously formed families of words.[44]

(4) Piaget has reported that one can "train" a child to perform beyond his normal level of cognitive maturity, but that this is largely useless and does not serve to accelerate cognitive growth. In some ingeniously-devised experiments, Bruner has raised some doubts concerning the accuracy of Piaget's observations in this matter.[45]

(5) It appears that Piaget has focused attention on a very particular selection of tasks—such as his famous "conservation" tasks in pouring water, etc.—and it is by no means clear that these tasks, taken together, form an adequate and appropriate set of "pegs" on which we can hang the mathematics curriculum. Many important aspects of mathematics remain untouched, and in the case of some others the analogies with Piaget's tasks may be misleading rather than illuminating.

(6) As Sheldon White of Harvard University has remarked, there are probably many adults who still operate on immature levels (in the Piaget classification scheme) *with respect to certain specific areas of functioning.*

(7) The kindest thing that can be said of Piaget's writing is that, at least in most English translations, it is obscure and frequently misunderstood.

(8) One who observes young children carefully cannot help but conclude that often a child's most creative, resourceful, imaginative behavior is *voluntarily offered by the child,* but *cannot necessarily be elicited by a* (perhaps heavy-handed) *"standardized" approach.* Piaget himself has written:

> It is possible that a difference in method is responsible for the difference between these two sets of results. Sometimes a child will react overcautiously to a standardized procedure and give an intermediary

[44] Vygotsky, *op. cit.,* pp. 52-53.

[45] Cf. Andrew T. Weil. "Harvard's Bruner and His Yeasty Ideas." *Harper's Magazine,* December 1964. pp. 86-89.

response, while a more flexible line of questioning would reveal that these responses did not entirely satisfy him, and that he is capable of going a little further. Above all, the important thing is to see whether the regression, be it apparent or real, is at the level of an intermediate stage, or of a final equilibration, for regression in this second case would obviously mean that in replying correctly in a previous session the subject was not yet really certain of his reasoning, and therefore was in fact at an intermediate stage.[46]

In serious child study, the perceptiveness and judgment of the observer always seem to be critical. It appears that the goal of "objectivity" can be achieved only by restricting attention to gross observations and highly oversimplified theories.

In any event, the Nuffield Project does suggest a possible new answer to how we shall arrange the mathematics curriculum, at least for children 12 years old or younger.

4. **"Mirror Cards."** Before leaving younger children, there are two further developments that merit attention. The first of these are the "Mirror Cards" developed at ESI by Marion Walter, of Harvard University. These cards have been described in some detail in an article by Marion Walter.[47]

For the present, suffice it to say that the Mirror Cards provide for a young child to use mirrors in various ways so as to complete various pictures in a symmetric fashion, so as to convince himself that certain pictures cannot be made in this way, and so forth. What is of special interest—besides the intriguing originality of this approach—is that, in the first place, the mathematical content is novel, consisting of symmetry and motion geometry of a type widely studied in England, on the continent, and in Russia, but contrasting with the synthetic Euclidean approach that has become traditional in the United States. Second, they represent an introduction of geometry at earlier grade levels than usual; third, they involve the use of manipulatable physical materials rather than a paper-and-pencil approach; fourth, they represent genuine exploration and discovery by the individual child, with a minimum of teacher interference and with no obligation to "get the right answer"—but rather for each child to do the best *he* can in devising ways to solve the various problems; and, finally, they are based upon a Bruner-Piaget learning paradigm that suggests that *first* the child "play around" and get experience exploring,

[46] Jean Piaget, in a Foreword to *Young Children's Thinking*, by Millie Almy, Edward Chittenden and Paula Miller. New York: Teachers College Press, Columbia University, 1966. p. iv. Reprinted by permission of the publisher. Copyright © 1966 by Teachers College, Columbia University.

[47] Marion Walter. "An Example of Informal Geometry: Mirror Cards." *The Arithmetic Teacher* 13 (6): 448-52; October 1966.

with verbal discussions being arranged to come later (if at all). This paradigm is especially marked in the case of these Mirror Cards, because there is no specific immediate payoff that is envisioned. Rather, it is assumed that, through this play, the child will build up his own personal mental imagery that should aid him in "assimilation" and "accommodation" in his future mathematical studies.

This general "enriching" or "readiness-building" aspect of *experience* is, of course, becoming one of the important themes of some—but by no means all—of the "new math" and "new science" projects, especially at the elementary school level.[48]

Since the conflict between different learning paradigms lies at the heart of a modern mathematics curriculum, we might repeat here a few paragraphs from the Nuffield Project materials, dealing with the question of what kinds of preverbal experience should *precede* any discussion of "3 + 2 = 5":

> . . . it is easy to drill children into writing a statement like "3 + 2 = 5" before they can really appreciate the meaning of the symbols. . . . Is it right to encourage children to write such statements before they can recognize the "fiveness" of five and identify the existence of five objects?
>
> Symbols should not be introduced before they are really meaningful and unambiguous. There are plenty of mathematical experiences which should come before "3 + 2 = 5" and which are perhaps more fundamental. By starting apparently more slowly, quicker progress based on understanding will follow later.[49]

5. **"Attribute Blocks."** As our last elementary school project, we might consider briefly the "attribute blocks" mentioned earlier. These appear to have been introduced by Vygotsky and other Russian students of cognition in an effort to study how children develop concepts. They have lately been assigned a different role: they are used by some "new math" projects as a teaching tool, usually in the primary grades.

While various projects use slightly different arrangements, we can think for the moment of wooden blocks, some shaped like triangles, some shaped like circles, some shaped like squares, and some shaped like parallelograms. They are of two sizes, which we can call "large" and "small." They may be painted yellow, green, red, or blue.

In introducing attribute blocks, the teacher might put out a large

[48] There is even some physiological research on rats that indicates that rats brought up in an "enriched" environment—filled with colors, shapes, exercise wheels, ladders to climb, etc.—develop brains which actually weigh more than those of rats brought up in a less "exciting" environment.

[49] *Beginnings* (1) . The Nuffield Mathematics Teaching Project. p. 1. Note that this is different from the booklet entitled *Beginnings* (1) .

red circle, then ask the children (who have not been allowed to see the complete set which the teacher is holding in an opaque container) what else they would like. A child might guess that perhaps there is a blue circle, so he might ask for that. As more and more of the blocks are displayed, the children acquire more information on what *other* blocks might still remain undisclosed.

There are many different "games" that are played with these attribute blocks, such as making rectangular arrays (ordered by row and column), arranging "chains" where each block differs from its predecessor in exactly one attribute, putting the blocks into sets A, B, $A \cap B$, and $(A \cup B)'$, etc.

Such play is thought to produce readiness for systematic thinking about mathematics, but there seems to be no convincing evidence, as yet, that it actually does.

The attribute blocks suggest another aspect of "new math" in the primary grades: if one *does* elect to introduce some version of the idea of *set, it is not necessary to relate this to the idea of "number."* Sets have many attributes, and (as we remarked earlier) the "number of elements in a set" is merely one among many possible attributes of sets that one might choose to study.[50]

6. **"Math Labs."** Turning (generally) to older children, some projects (e.g., the Madison Project) and many schools are building "math labs." The rationale is something like this:

a. Following Piaget, it is assumed that actual perception and actual active manipulation of physical materials contribute to concept formation, for some students if not all students;

b. The evidence seems to indicate (rather strongly) student preference for this method of learning;

c. Watching a child manipulate physical materials often gives the teacher deeper insight into how the child is thinking about a task than can be obtained by purely verbal methods;

d. Traditional verbal methods seemed to create a superficial rote learning that did not seem to help the child when—as in shop or lab—he was confronted with tasks involving real objects;

e. Using actual physical objects necessarily challenges the "one

[50] One of the simplest and most elegant versions of "attribute blocks" and "attribute games" is available from Educational Services, Incorporated, 55 Chapel St., Newton, Massachusetts 02158; a rather different and vastly more complicated discussion of the subject is given in: Z. P. Dienes and E. W. Golding, *Learning Logic, Logical Games* (Herder and Herder, "Modern Mathematics Experience Programme" series, 232 Madison Ave., New York, N.Y. 10016, 1966). Some observers believe that Dienes' use of attribute blocks is too complicated to be appropriate for work with young children.

right answer" syndrome that afflicts many children and teachers (and which we shall discuss further presently);

f. "Math labs" create a desirable classroom social setting for individualizing instruction.

7. **The Nova School Program.** We turn now to what is perhaps the most exciting and the most frustrating vista on the entire "new mathematics" landscape—namely, the program in high school mathematics created by Burt Kaufman at the Nova School, a public high school of Broward County, Fort Lauderdale, Florida.

This program has been visited and very carefully scrutinized by eminent mathematicians and educators from most of the major universities in the United States, as well as by representatives of state and federal agencies and private foundations (and, incidentally, also by European mathematicians and educators). It has been declared almost unanimously to be the highest quality pre-college mathematics program in the United States. In the opinion of the present writer this was no exaggeration—perhaps no one has visited every school in the nation, but if there has been another program this good it would surely have been difficult to keep it a secret.

The Nova program was a four-track program; we shall consider here the top track (for, roughly, the most able 25 percent of the students).

What one had at Nova was a serendipitous combination of an exceptionally imaginative school situation with an exceptionally gifted mathematics teacher (and a hard-working and devoted cooperating faculty). Nova School was designed by a group of forward-looking and courageous men, both on the Board of Education and in the administration. It seems impossible to answer the question of "whose idea it all was," but one of the chief designers of the Nova School was unquestionably Arthur B. Wolfe. The Assistant Director for Instruction has been James Smith, and the Broward County Superintendent of Schools is Myron Ashmore. As mentioned earlier, the head of the mathematics department was Burt Kaufman, an exceptionally fine mathematician and a devoted teacher.

In an age when some schools spend $1200 per year per student, Nova earned its reputation for the finest mathematics program in the United States with a program that cost $300 per year per student.

Students in Broward County were enrolled at Nova only if they requested it; vacancies were filled on a "first come first served" basis. This enabled Nova to gain a degree of initiative and freedom unusual among United States public schools—although Nova is, in fact, a tax-supported public school in the Broward County system.

Nova built its economical but excellent program in part by what they refer to as "the reallocation of money and resources."

Nova takes advantage of its good climate by using the outdoors for as many activities as possible. Nova provided virtually no food service—how students obtained food was a parental-student responsibility. Nova provided no transportation—getting to school and back was a parental-student responsibility. Nova virtually eliminated vacations—Nova students attend school eleven months a year. Nova eliminated baby-sitting and custodial care: as in a college, if a student has no class between 10 a.m. and 11 a.m., how he spends his time is his responsibility. He may study, work in a learning center, participate in a student-organized seminar discussion, or wander around outdoors. For Nova faculty there is no such thing as lunchroom duty, study-hall duty, school-bus duty, and so on.

The Nova school is not arranged by grade level. Students work together if and only if they are ready to approach a common problem in a similar fashion. Time is never used as a measure of instruction at Nova: teachers are not expected to finish a book (or even a unit) by some arbitrary date. After the one-month summer vacation they merely resume where they left off at the beginning of the vacation.

The building complex has small conversation or seminar rooms, where students work together without a teacher; it has learning centers equipped with modern A-V facilities, including video-tape and closed-circuit TV. It has an office for each teacher, and secretarial help for teachers. Many rooms have movable partitions, and all rooms have overhead projectors and similar equipment.

In a large-group team teaching situation, a teacher at the front of the room may be going over the homework, one problem at a time, using an overhead projector and a very superior P.A. sound system. The advantage of the P.A. system (and the excellent room acoustics) is that, somewhat as in an airline terminal, one can hear the P.A. if one wishes, or else one can ignore the P.A. and can participate in local small-group discussions without interfering with other people's chance to hear the P.A. Indeed, this is what usually happens; various small groups work together, and pause to watch the teacher only when he reaches a problem which they wish to hear discussed. While one teacher is using the overhead projector and the P.A. sound system, one or more other teachers circulate among the small groups of students offering individualized attention.

One central fact has been the successful shift of responsibility from the faculty to the students. Each student is responsible for his own education, and accepts the responsibility. Both the casual visitor and the long-term investigator cannot help but be impressed by the remarkably mature

and responsible attitude of the students at Nova. The old claim that children behave the way we *expect* them to behave seems to be borne out—and Nova expects the students to behave in a responsible way. (One even sees children working in the science lab with no teacher in the room—although the inside walls are glass, so any problems could be seen immediately.)

For the top-track students, Nova *virtually abolished high school mathematics.* The simple skills of factoring polynomials, graphing functions, solving triangles, etc., which normally constitute virtually the entire high school mathematics program are merely assigned to the top-track students for outside study. School time is not "wasted" on such material. Eighth graders (using chronological age to decide who should be called "eighth graders") may begin the study of a high quality modern program of *university-level mathematics.*

The program works. One comparison found the Nova 8th and 9th grade students *ahead* of the students in one of the nation's leading universities.

Such a triumph is exhilarating to observe. The visitor comes away feeling that he has seen the future. Soon *all* schools *must* be like this!

But they will not be. The diffusion of innovation in education is not so simple. Nor could "ordinary" teachers cope with the Nova program—they could not. (This, in itself, is a devastating indictment of the short-sighted way we have arranged teacher education programs.)

Nova exhibits all of the classical problems of superb excellence and unsurpassed innovation, even to the fact that Burt Kaufman and his chief co-worker, Joseph Karmos, came to a position of disagreement with the administration and left, to attempt to rebuild the Nova program at the Laboratory School of Southern Illinois University at Carbondale, Illinois. Roger Robinson, Principal of the University School, Robert W. MacVicar, Vice-President for Academic Affairs, Dean Elmer Clark of the School of Education, John W. Olmstead, Chairman of the University Mathematics Department, and Wade Robinson of the St. Louis Regional Laboratory are cooperating in trying to build a Nova-type program at Carbondale. Whether the future will contain *two* Novas, one in Florida and one in Illinois, or none at all, remains to be seen. (Incidentally, the Carbondale school is replacing the "unit" with the "activity package" for still more effective individualization of instruction,[51] and is planning ex-

[51] Cf. Garrett R. Foster, Burt A. Kaufman and William M. Fitzgerald. *A First Step Towards the Implementation of the Cambridge Mathematics Curriculum in a K-12 Ungraded School.* Report submitted to the Commissioner of Education, U.S. Office of Education, in relation to Cooperative Research Project No. S-405 (Florida State University, Tallahassee, Florida, 1966).

tensive use of computer-assisted learning equipment.) The MIT- and Harvard-dominated Cambridge Conference on School Mathematics has volunteered its assistance in building the new program at Carbondale.

If Nova (and Carbondale) succeed in establishing a pattern, all of today's high school mathematics is obsolete as material for study on school time by upper-quartile students. This is the most revolutionary development on the present horizon. (There is one exception: Nova and Carbondale have retained one piece of the traditional high school program: Euclidean synthetic geometry. However, they use as a student text the book which Edwin Moise wrote for study by teachers,[52] so even here it might be said that the "traditional" content has been discarded.)

The issue of whether isolated instances of superb quality and breathtaking innovativeness can have significant effect on United States education is central to the entire effort to improve our schools and our education. Have the availability of film and video-tape, and the modern achievements in transportation and communication, made it possible for a Nova to have nation-wide influence? It is no exaggeration to say that the future of American education depends upon the answer to this question.[53]

Writing in the *New York Times,* Fred Hechinger stated that exciting new programs depend upon a committed faculty whose members work harder from their sense of commitment than most men will do from a mere need to earn money. Once these original innovators depart, the program necessarily becomes far more costly, since two or three must be hired to do what was formerly the work of one dedicated person. Hechinger was generalizing from New York's *Higher Horizons* program, but his description might apply equally to Nova. Both administration and teachers *regularly* worked until 2 a.m., *and showed up for work early the next day*! This—as usual when men "catch fire" with a new idea—is also part of the secret of how Nova was able to make $300 per year per student accomplish so much. Yet it suggests that some of the hard facts of economic life would necessarily catch up with Nova sooner or later.

Nova also suggests another important aspect of the "new mathematics" movement: genuine progress is only possible when innovative administrators work together with innovative mathematicians and teachers . . . and when they do, all aspects of school programming become in-

[52] E. E. Moise. *Elementary Geometry from an Advanced Standpoint.* Palo Alto, California: Addison-Wesley Publishing Company, 1963.

[53] Reading Cremin's *Tranformation of the School* suggests that 50 years ago isolated exemplary practice could not be propagated rapidly and widely. That, however, was before the age of jet airliners and satellite TV.

volved: scheduling, audio-visual facilities, even school architecture. Without such an all-out effort, the "curriculum evolution" movement of the 1950's and 1960's will not produce significant improvement in education in the United States.

8. **The Advanced Placement Program.** We turn now to a curriculum innovation of a different sort: The Advanced Placement Program of the College Entrance Examination Board. This is not usually thought of as a "new mathematics" program, but in fact it represents one of the most important changes in high school mathematics in the past two decades. What is involved is, in effect, moving calculus and analytic geometry from the freshman year at major universities into grade twelve at a growing number of high schools. The importance of this program arises from several aspects.

a. Obviously, for some bright college-bound students this will save both time and money from their pre-professional education. This is important in view of the increasingly high cost—both in time and in money—of becoming a physician, an engineer, a teacher, a social worker, a scientist, etc.

b. What is perhaps even more important, it gives high school faculty an opportunity to teach a larger portion of mathematics; this means that it becomes easier to recruit mathematically-talented faculty, and that teachers, once recruited, can gain a better comprehension of the true nature of mathematics by having more extensive contact with more mathematics.

A theme which will recur below is spelled out in detail in Callahan's *Cult of Efficiency*[54]: The explosive expansion of United States education has caused an unmistakable shortage of subject-matter competence in real depth. There were not enough scholar-teachers to go around, the expansion of college education and industrial employment has drained away a very large fraction of those who did exist, and the "cult of efficiency" and "business management"[55] orientation of public schools created an environment unfavorable for acquiring and developing scholars of significant depth. The accomplishment of significant improvement in school

[54] Raymond Callahan, *loc. cit.*

[55] It should be emphasized that the "business management" which Callahan shows invading our schools in the early part of this century was *not* 1960-style business management from research-minded industries like Xerox, IBM, Westinghouse, Monsanto, or Boeing. At the present time, such industries may offer a more suitable home for scholars than our schools do. But—as I hope the present survey makes clear—our schools need scholars quite as much as industry does. With a few exceptions, however, the schools have failed to recognize this need.

mathematics can be carried out only to the extent that we can alleviate this shortage of "scholars-in-depth" in our schools.

From the point of view of the job description of the high school teacher, the introduction of calculus into high school is similar to the increased use of teachers in after-school teaching at community colleges, at universities, and in teacher education programs: all of these provide the teacher with important opportunities for growth *as a teacher*, and *as a scholar*. In this regard they are quite different from those pressures which tend to remove the teacher from the classroom, and to channel him into guidance, supervision or administration.

From the point of view of the upper-quartile student, the Advanced Placement Program resembles to some extent the Nova program: both relegate traditional "high school mathematics" to a smaller share of the student's time, and get on to more advanced matters sooner than the traditional program did.

In the past few years this tendency has been growing, and some schools now use advanced placement mathematics as early as grade eleven. This represents an advance of at least two years over the traditional timetable.[56]

[56]Some idea of the size (and, more important, the rate of growth) of the CEEB Advanced Placement Program can be gained from the following table:

Year	Schools	Students taking examinations	Examinations taken	Colleges entered
1955-56	104	1,229	2,199	130
1956-57	212	2,068	3,772	201
1957-58	335	3,715	6,800	279
1958-59	560	5,862	8,265	391
1959-60	890	10,531	14,158	567
1960-61	1,126	13,283	17,603	617
1961-62	1,358	16,255	21,451	683
1962-63	1,681	21,769	28,762	765
1963-64	2,086	28,874	37,829	888
1964-65	2,369	34,278	45,110	994

This table is taken from p. 15 of the College Entrance Examination Board booklet entitled *Advanced Placement Program: 1966-68 Course Descriptions*. (Available from College Entrance Examination Board, Publications Order Office, Box 592, Princeton, N.J., 08540.) According to the UNESCO report *University Instruction in Mathematics: A Comparative Survey of Curricula and Methods in the Universities of Czechoslovakia, Federal Republic of Germany, France, Japan, Poland, United Kingdom, U.S.A., U.S.S.R.* (1966) about 30 percent of the students entering Harvard University as freshmen have already completed a full year of college-level calculus and analytic geometry in high school (p. 165). This UNESCO volume also shows clearly that "what is going on in mathematics" is a common problem shared by all of the technologically-advanced nations today.

Reprinted with permission from *Advanced Placement Program: 1966-68 Course Descriptions*, published in 1966 by the College Entrance Examination Board, New York.

9. **Introducing New Topics and New Courses.** At the secondary school level, especially, the stereotype of "new math" is that it is concerned with introducing into the curriculum various new topics and new courses. This is, indeed, both true and important.

"Pure" mathematics—that is to say, the internal or intrinsic part of mathematics, removed from any context of applications to physics, engineering, etc.—was completely revised and reorganized in the 19th and early 20th centuries, under the influence of such mathematicians as Cayley, Gauss, Riemann, Cauchy, Weierstrass, J. W. Gibbs, Weyl, Poincare, Von Neumann, Hilbert, and many others. After this reorganization, mathematics was no longer the same subject that it had been in the days of Newton, Euler, Cardano, Viète, and Descartes. Yet most of this reorganization did not reach college undergraduate programs, and virtually none of it was reflected in secondary school programs.

Ninth-grade algebra has continued to be studied in the style of Euler (1707-1783); tenth-grade geometry has continued to be studied in the style of Euclid (who lived around 300 B.C.; an English translation of Euclid's *Elements* dates from 1570 A.D.); calculus has continued to be studied (usually in college) in the style of Newton (1642-1727) and Leibnitz (1646-1716); and such "modern" topics as vectors (nineteenth and twentieth centuries), matrices (developed by Arthur Cayley in 1857), non-Euclidean geometries (developed by Lobachevsky, Bolyai, Gauss, and Riemann in the first half of the nineteenth century), statistics (late nineteenth century, and twentieth century), and mathematical logic (twentieth century) have not usually been studied at all until one has reached the last two years of college, or graduate school.

Yet if mathematics was completely changed *internally,* this amounts to nothing compared to the changes in the *applications* of mathematics to science and technology. The advent of high speed electronic digital computers constitutes an industrial revolution second to no other. Even the question of *who uses mathematics* has a different answer nowadays. In 1900, one could answer: physicists, astronomers, chemists, actuaries, and engineers. In the 1960's, the answer can be inferred from such facts as the observation of colleges *where the computer laboratory is located in the school of education.* Nowadays one would have to answer: biologists, economists, psychologists, physicians, geologists, metallurgists, meteorologists, military decision makers, business decision makers, educators, historians, linguistics experts, sociologists, epidemiologists, lawyers—and, as before, all of those who work within mathematics itself, or in any part of the physical sciences and technology. ("Physical science and technology" have *themselves* changed so evidently that it hardly requires commenting

on the difference between color TV, space satellites, jet aircraft, and super highways, vs. the technology of 1900.)

Evidence of new topics at the secondary school level (and in one or two cases, even at the elementary school level) can be gained from the following list of recent books:

Asger Aaboe. *Episodes from the Early History of Mathematics.* New York: Random House, 1964. This—and all the other volumes in the SMSG-inspired "New Mathematical Library"—are intended to give high school students an opportunity to read about various aspects of mathematics on their own, outside of class.

Carl B. Allendoerfer and Cletus O. Oakley. *Principles of Mathematics.* New York: McGraw-Hill Book Company, Inc., 1955, 1963, often used in grade 12.

C. A. R. Bailey. *Sets and Logic* (2 vols.). London, England: Edward Arnold, 1964.

Garrett Birkhoff and Saunders MacLane. *A Survey of Modern Algebra.* New York: The Macmillan Company, 1944. This was probably the earliest book to "spread the word" on the reorganization of the algebraic part of mathematics, at the college level. Today much of this content has moved into the high school.

Leonard M. Blumenthal. *A Modern View of Geometry.* San Francisco: W. H. Freeman & Co., 1961.

Richard Courant and H. Robbins. *What Is Mathematics?* New York: Oxford University Press, Inc., 1941. Not a really recent book, but an excellent, relevant and "modern" one that was ahead of its time, and has now become a classic.

Ralph Crouch and David Beckman. *Linear Algebra.* Glenview, Ill.: Scott, Foresman & Co., 1965. This book is intended to serve "as a transition from the mathematics of high school to the mathematics of college."

Philip J. Davis. *The Mathematics of Matrices. A First Book of Matrix Theory and Linear Algebra.* Waltham, Mass.: Blaisdell Publishing Company, 1965. This book has been used successfully in grade 12.

Robert B. Davis. *Explorations in Mathematics.* Palo Alto, Calif.: Addison-Wesley Publishing Company, 1966. This book deals in an introductory way with logic, statistics and matrices, and is aimed at grades 5 through 9.

William S. Dorn and Herbert J. Greenberg. *Mathematics and Computing.* Part 1 and Part 2, Preliminary edition. New York: John Wiley & Sons, Inc., 1965.

Howard Eves and Carroll V. Newsom. *An Introduction to the Foundations and Fundamental Concepts of Mathematics.* New York: Holt, Rinehart and Winston, Inc., 1958. One of the earliest "modern mathematics" books, and still one of the most important.

Howard Eves. *An Introduction to the History of Mathematics.* Revised

edition. New York: Holt, Rinehart and Winston, Inc., 1964. Can be used in grades 9-12, and possibly earlier, as well as in teacher education.

T. J. Fletcher. *Some Lessons in Mathematics*. New York: Cambridge University Press, 1965. This volume includes some computer-related mathematics, a modern approach to some algebra (groups, rings, fields), logic, matrices, and some mathematics used in modern business decision-making procedures (e.g., linear programming). This book is suitable at any secondary grade level, if not earlier.

K. L. Gardner. *Discovering Modern Algebra*. New York: Oxford University Press, Inc., 1966.

Shirley Hill and Patrick Suppes. *First Course in Mathematical Logic*. Waltham, Mass.: Blaisdell Publishing Company, 1964. This is a course in logic actually used with sixth graders!

Banesh Hoffman. *About Vectors*. Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1966. This work is possibly suitable for grades 10, 11 or 12 (or later).

James A. Hummel. *Vectors*. Palo Alto, Calif.: Addison-Wesley Publishing Company, 1965. This book is suitable for grades 10, 11 or 12.

John G. Kemeny, J. Laurie Snell and Gerald L. Thompson. *Introduction to Finite Mathematics*. Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1957. This work includes logic, probability, and matrix algebra, and has been widely used at the 12th grade level, as well as in teacher education.

John G. Kemeny and J. Laurie Snell. *Mathematical Models in the Social Sciences*. Waltham, Mass.: Blaisdell Publishing Company, 1962. This is a college-level book.

F. B. Lovis. *Computers* (2 vols.). London, England: Edward Arnold, 1964.

D. E. Mansfield and M. Bruckheimer. *Major Topics in Modern Mathematics*. New York: Harcourt, Brace & World, Inc., 1965.

Geoffrey Matthews. *Matrices* (2 vols.). London, England: Edward Arnold, 1964.

Herbert Meschkowski. *Ways of Thought of Great Mathematicians*. San Francisco: Holden-Day, Inc., 1964.

F. Mosteller, R. E. K. Rourke and G. B. Thomas, Jr. *Probability: A First Course*. Palo Alto, Calif.: Addison-Wesley Publishing Company, 1961. This book is suitable for grade 12 (or possibly earlier).

Howard M. Nahikian. *Topics in Modern Mathematics*. New York: The Macmillan Company, 1966. This is perhaps an especially eye-opening book for the nonmathematician who wishes to get an overview of "what new math is." It includes valuable references to historical matters. The publisher and author suggest that this book is appropriate for use in high school.

Moses Richardson. *Fundamentals of Mathematics*. New York: The Macmillan Company, 1939, 1966. Another of the very early books on "modern mathematics."

School Mathematics Study Group (SMSG). *Mathematics and Living Things.* Available from the School Mathematics Study Group, Stanford University, Stanford, California, 1965.

Peter Wolff. *Breakthroughs in Mathematics.* New York: Signet Science Library, 1964. (Paperback.)

What effect has all of this effort in writing and publishing had on school mathematics in the United States? From all presently available evidence, its effect is relatively slight. Most of the high school mathematics curricula seem—once one looks beneath the shiny new surface—to be about the same as they were before the alleged "revolution" began.

Indeed, as one considers the forces for change that exist today, and the actual curriculum and instruction that can be observed in operation in the vast majority of schools, one is forced to agree with William M. Alexander that the curriculum has become frozen into a rigid, inflexible form that is no longer responsive to the forces that seek to shape it. Professor Alexander's analogy is apt, and raises the question of whether the curriculum can become "unfrozen" and assume a fluid form that can adapt to modern needs, or whether—and ice behaves this way, too—it will be shattered into a fine spray of minute pieces. Reading Cremin's volume provides powerful evidence that there has been no recent revolution—the curriculum has changed less in the past several decades than it did in, say, the 1920's and 1930's. Our *society* is changing much faster, but our *curriculum* is changing more slowly. This does not bode well for the decades ahead.

10. **Improving the Content in General Mathematics.** About this I shall say little, except to point out that *most* attempts to improve "general mathematics" (or secondary school mathematics for low achievers) tend to be unimaginative. The same content is taught in more-or-less the same old way, but with added fervor and determination. This applies (unfortunately) to too much of the work of the Job Corps Training Centers. By contrast, a *few* attempts have been far more creative and imaginative. Most of these "imaginative" programs have varied the content, often trying a wide variety of mathematical topics in the hope of striking some spark of interest; many have also used physical materials, ranging from Cuisenaire rods to desk calculators and even to electronic computers. Some have used teaching machines. Some have manipulated the classroom environment (in some cases shifting it away from its usual middle-class tone). Some have sought to teach "harder" material rather than "easier" topics. Some have abandoned traditional sequences of topics—for example, by moving on to analytic geometry and algebra rather than continuing the uphill battle with arithmetic. In some cases, the mathematics

is tied to vocational courses in the operation of a television studio or courses in wiring and operating digital computers.

This, too, is part of the "new math"; it also bears, at least in the more imaginative examples, the unmistakable brand of "progressive education—1967 style."

11. **Experimenting with Textbook Style.** "Programmed Learning" received so much attention a few years ago (while achieving little by way of educational improvement) that it masked a perhaps more important experimentation in textbook style that began even earlier, and had many elements in common with programmed learning.

Some exceptionally fine examples are contained in the small booklet, *Formulas, Graphs and Patterns,* Unit I of the series, *Experiences in Mathematical Discovery,* published by the National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington, D. C. 20036. This series is intended as an imaginative revision of "general mathematics," and is excellent for the purpose, but in fact it could well be used with bright elementary school children, and elsewhere, since what it actually is is an exceptionally clear and well thought out introduction to basic mathematical concepts such as Cartesian coordinates, the arithmetic of signed numbers, etc.

This "text" largely avoids expository writing, which tends to cast the student in so passive a role that he fails to become involved at all. In its place the text uses a sequence of questions. Many of these questions string together to form a running dialogue between two boys (although other formats are used as well). Here is an example:

1. It is decided that John, the older of the two boys, will take the first turn. John is allowed three "shots" at David's battleship.
   First shot: (5 to the right, 3 up)
   Second shot: (7 to the right, 1 up)
   Third shot: (4 to the right, 4 up)
   David looks at his chart and tells John that he missed on all three shots. Do you agree?

2. Now it is David's turn to shoot at John's battleship. (It is agreed that to sink a battleship one must make three successful shots.)
   First shot: (2 to the right, 1 up)
   Second shot: (4 to the right, 2 up)
   Third shot: (1 to the right, 1 up)
   John admits that David has made one hit and that his battleship is damaged. The boys agree that it is not necessary to tell which shot has been successful. Do you agree that there has been one successful shot? If so, which one?

3. It is John's turn again.
   First shot: (4,5)
   Second shot: (5,4)
   Third shot: (2,2)
   The boys agree to use the symbol (4,5) as an abbreviation for (4 to the right, 5 up).
   a. Does (4,5) indicate the same position as (5,4)?
   b. Does the *order* of the numbers named in the two symbols make a difference in the position of the point that the symbols represent? A pair of numbers arranged in a definite order, such as (4,5) or (5,4) is called an *ordered pair* of numbers. We call the number that is named first the *first number* of the ordered pair and the number that is named second the *second number* of the ordered pair.
   c. Does John have a hit? If so, which shot was a hit?
4. For his next turn David calls the following ordered pairs of numbers: (1,7), (7,1), (7,7). John is surprised. He remarks that David has just wasted three shots.
   a. Do you agree with John? Why?
   b. How can the boys keep a record of the shots they have taken?

One important advantage which many observers see in this style is that many statements are attributed to children, and may in fact be correct or may be wrong. Thus the reader is kept on his toes, and must read everything critically. He is led away from the usual but intellectually disastrous procedure of "believing everything he reads in a book."[57]

12. **"Discovery" Learning.** Not quite all of the "new mathematics" programs claim that they use "discovery learning" or "discovery teaching." What—if anything—this means requires considerable study. There are many different interpretations of what "discovery" means.

This is an important example of a very important general problem: in the "new math revolution," words are given so many *different* meanings that they end up by having almost no meaning at all.

Perhaps the main thing that the reader can learn from this entire booklet is the fact that real progress in education for the next few decades would seem to depend upon a more profound level of discussion, and a more profound level of discussion almost certainly requires more agreement on what we mean by various words. We must stop saying "the new math." Which version of "new math"? One which is more abstract than "the old math" (whatever that was), or one which is more concrete? A

[57] The present author has also experimented with textbook "style" in several somewhat similar fashions, in the books, *Discovery in Mathematics* and *Explorations in Mathematics*, published by Addison-Wesley Publishing Company, Palo Alto, California.

version which uses physical materials, or a version which uses only paper and pencil? A version which is "harder" (another misleading word, since it sometimes refers to difficulty, sometimes to unpleasantness, sometimes to a degree of novelty, and sometimes to level of sophistication) than "old math," or a version that is "easier" than "old math"? Every word on the following list has come to be used in flagrantly different (and irreconcilable) ways:

new math
discovery
sets
abstract
concrete
achievement test
a supplementary program
an enrichment program
creativity
simplicity
geometry
algebra.

In the case of "discovery," consider these questions and differing interpretations:

a. One project directs a child to pour two pints of water into a quart jar, and tells him he has "discovered" that 2 pints are equivalent to 1 quart. Many "modern mathematics" specialists consider this to be a horrible example of what "discovery" should *not* mean. They claim the child merely did what he was told, that there was no exploration or "playing around" involved, and since there was no exploration there can have been no discovery. These critics argue that what the child did here was either "to observe that . . ." or else "to confirm that . . .". But—they claim—*confirmation* is not at all the same thing as *discovery*.

b. What is the essence of "discovery"? For some people, it lies in *allowing the child freedom to explore*. For others, it lies in *not verbalizing the "patterns" that the child observes*—somewhat like "not spoiling a piece of music by discussing it too much." For some, this verbalization is acceptable (or even desirable) provided it does not take place immediately—the child should be allowed to apprehend the newly perceived pattern for itself alone, unencumbered by descriptive words, for at least a few days, hours or weeks. After that it is all right to talk about it. For some observers, the reason for this restraint is that the teacher does not want to detract from the child's "discovery" by saying

(in effect) "Oh, yes; everyone knows *that*!" This is perhaps a matter of *respect for the child*, or possibly even a question of good manners. (One might compare this to a father's obligation to be frightened when his 3-year-old son becomes a ferocious tiger and starts to attack him. If father is *not* frightened, he has in effect said that his son's ferocious abilities do not really amount to much.)

Other observers say it is all right for *children* to verbalize the pattern provided the *teacher* does not—this procedure (as in the preceding section on textbook style) is considered valuable on the grounds that, since children are often wrong, the listener is obligated to listen critically, and to reject part of what he hears; he cannot form a habit of uncritical acceptance of everything people tell him.

Other observers reverse this: they say it is all right for the *teacher* to verbalize patterns, provided the *children* do not. These observers argue that, since children rarely "say it correctly," the other children in the class are perpetually bombarded with incorrectly-stated "facts."

Still other observers argue that the inexact language of the child has meaning for the child. It belongs to his world. By contrast, the "sophisticated" language of the adult is meaningless, and leads the child into rote repetition of what he hears.

Others argue that *all learning* is necessarily "learning by discovery" —no matter whether the *experience* comes first, or the *verbal discussion* does, no real *learning* has occurred until the two are correctly paired together. If the discussion comes first, then—perhaps years later—the discovery occurs when you have a relevant experience, and suddenly get that 'Aha!' feeling: *"now* I know what they were talking about when they said . . . ."

c. Experimental studies of "discovery" are limited by a lack of agreement as to the advantages that are believed to stem from "discovery learning." For some, the advantage is believed to lie in the direction of making school more interesting, making school a more desirable place to be. To test this, one would need to observe school dropout rates, or the proportion of students reading intellectual books in later years, or something of the sort. And—in performing the experiment—one would have to be careful to ensure that school *actually was* "a desirable place to be" and that intellectual questions did in fact appear in an attractive light.

For some people, "discovery" teaching is in effect *good theater*, and is designed to attract the child's attention and to hold it.

For others, "discovery" is believed to help avoid rote or superficial learning, and to promote deeper, more meaningful learning.

There are many other alleged "values" in discovery teaching; in each case, an experimental study would have to be carefully executed in order to be sure that the relevant value was actually present, and in testing one would have to be sure that one was testing for the expected outcomes.

It is by no means true that "discovery" teaching is necessarily aimed at higher scores on paper-and-pencil achievement tests.[58]

Robert Karplus of Berkeley has identified two aspects of human thought that may be relevant to "discovery": one is a tolerance for unsettled questions, tentative answers, and open-endedness, as opposed to a need for finality and closure (at the price of saying, in effect, "My mind's made up; don't confuse me with the facts!"); the other is a tendency to identify inconsistencies within one's own thinking, and to attempt to resolve them, as opposed to a tendency to keep different ideas in separate compartments of one's mind and to allow as few confrontations as possible (as one does when one professes Christianity on Sunday and does not practice it on Monday through Saturday, or when one extols "democracy" but ignores the situation of many Negro Americans).

One might argue that such matters are not the concern of *mathematics* curricula. Such an argument would ignore the realities that shape human behavior. We do not usually teach what we say we set out to teach. Consider the following sentences, written by one of the greatest of today's computer experts, in an article on computers *that was not intended to deal with education*. Quite clearly, as he wrote about computers, Professor Strachey found himself thinking about people, and as he thought about people, he found himself thinking about education:

> In the early days of computer programming—say 15 years ago—mathematicians used to think that by taking sufficient care they would be able to write programs that were correct. Greatly to their surprise and chagrin, they found that this was not the case and that with rare exceptions the programs as written contained numerous errors. The process of discovering, locating and correcting these errors proved to be one of major difficulty, often taking considerably longer than writing the program in the first place and using a great deal of machine time.
>
> Although programming techniques have improved immensely since the early days, the process of finding and correcting errors in programs—known, graphically if inelegantly, as "debugging"—still remains a most difficult, confused and unsatisfactory operation. The chief impact of this state of affairs is psychological. Although we are all happy to pay lip service to

[58] One interesting reference on "discovery" is the Nuffield Mathematics Project pamphlet entitled, *A Look Ahead*.

the adage that to err is human, most of us like to make a small private reservation about our own performance on special occasions when we really try. It is somewhat deflating to be shown publicly and incontrovertibly by a machine that even when we do try, we in fact make just as many mistakes as other people. If your pride cannot recover from this blow, you will never make a programmer.

It is not, in fact, in the nature of human beings to be perfectly accurate, and it is unrealistic to believe they ever will be. The only reasonable way to get a program right is to assume that it will at first contain errors and take steps to discover these and correct them. This attitude is quite familiar to anyone who has been in contact with the planning of any large-scale operation, but it is completely strange to most people who have not.

The trouble, I think, is that so many educational processes put a high premium on getting the correct answer the first time. If you give the wrong answer to an examination question, you lose your mark and that is the end of the matter. If you make a mistake in writing your program—or, indeed, in many other situations in life outside a classroom—it is by no means a catastrophe: you do, however, have to find your error and put it right. Maybe it would be better if more academic teaching adopted this attitude also.[59]

Surely it is only reasonable to say, on present evidence, that people do seem to be shaped by their education, and that many non-mathematical aspects of behavior are presumably influenced by (among other things) the *manner* in which one encounters mathematics, and by the types of experiences one accumulates, whether in the mathematics classroom or elsewhere.

13. **Modifying Testing Procedures and Rationale.** People do not agree on what they mean by "tests" or "evaluation." We have already encountered Professor Strachey's suggestion that one be allowed a second try, or even a third or a fourth. He suggests, in effect, that we "test" the ability to arrive at a satisfactory result, given the help of an intelligent critic who will point out deficiencies in our work (which is what the computer does in Strachey's example).

Some schools have experimented with allowing students to take an "exam" as a group effort, with dissenters being allowed to file minority reports.

If one analyzes discussions on tests, it becomes clear that some people regard a test as a hurdle, and accept this as the goal of instruction. They

[59] Christopher Strachey. "System Analysis and Programming." *Scientific American,* September 1966. pp. 118-20. Copyright © 1966 by Scientific American, Inc. All rights reserved. A more extreme suggestion of this sort is proposed by Lewis Anthony Dexter in *The Tryanny of Schooling*, New York: Basic Books, Inc. 1964.

argue that if a student gets over the hurdle—by any means short of actually cheating—then this accomplishment *by itself* proves that the goal of the instruction was realized. Those who hold this opinion see nothing wrong in special study sessions aimed specifically at helping the student do well on the particular test in question. Indeed, many of them advocate precisely this.

Others hold that education aims at preparing a student to cope with an unknown future, and that a "test" is a *sample* of possibly relevant behaviors drawn from some rough idea of what the future may hold. Those who view tests this way believe that one can make no inferences about the student's future behavior in an unknown environment unless there is an element of randomness in the test items—the student must not be drilled in advance for the specific items on a specific test. On the contrary, he must enter the test as he enters the future—not being at all sure of what is in store for him. "Drilling" a student specifically for the test is anathema to this group.

These (and other) differences are not mere quibbling. Quite the opposite: there is so much disagreement on *what* one should "test" and *how* one should do it that most thinking about tests is superficial and most discussion is unproductive. (As one example, many discussions ignore the effect that tests have on the way that teachers teach—but this effect is extremely apparent and very important.)

Some observers are more interested in how a student goes about attacking a problem than they are in what "answer" he gets. Some prefer open-ended items in which there *is* no clear-cut final "answer." Some wish to test how a student uses mathematics *when there is no pressure on him to use mathematics at all.* Some are especially interested in "elegance" or "style." Some will use only paper-and-pencil tests, while others want to observe how the student uses mathematics when he is dealing with actual physical objects. Some want to observe the student's ability *to pose the question,* rather than (or in addition to) his ability *to answer the question once it has been stated.*[60]

Little has been done to study, for example, the effect on teacher be-

[60] I am indebted to Professor McShane of the University of Virginia for the following mathematics problem, which I state in its entirety:

A pile of coal catches on fire.

The task here, quite evidently, is not so much to find the answer, but rather *to find the problem.* In this same connection, J. Richard Suchman has focused attention on the process by which a student inquires into a subject, and the processes by which he develops a "theoretical" explanation and tests it for effectiveness. Some recent examinations for doctors and nurses focus on the questions which the doctor poses (by requesting lab tests, etc.).

havior that is produced by these various procedures and philosophies in "testing."

If one pursues such discussions to any depth, one finds quite clearly that it is nonsense to say that the superiority of some particular curriculum has been "proved by the results on objective tests."

The matter is made yet worse by the fact that different curricula use different definitions and different notations; that answers that are "correct" in the context of one curriculum are often "wrong" in the context of some other curriculum[61]; and that there is no agreement on a body of common knowledge that should be learned by every student.

Even more serious problems arise when a curriculum (often for very good reasons indeed) chooses to delay some topic, and to study it at a *later* point in the curriculum than is usual. The reasons for such a step are often very forceful; consider these examples:

Biology used to be the "easiest" of the "sciences," since it used to involve little or no mathematics, and little or no chemistry or physics. Nowadays biology is entirely different, with much interest in areas of biology that *do* depend upon mathematics, physics, and chemistry. Hence, many curriculum workers would like to move biology from its present location (usually in grades 9 or 10) to a new location in grade 12.

Geometry is traditionally in grade 10 (in most, but not quite all, schools). As one refines the study of geometry, it becomes more complicated. One project seriously considered moving geometry to grade 12, but ultimately abandoned the idea as impractical in terms of school politics.

Dividing one fraction by another is a complicated process, whose *meaning* is understood by few children, few "educated" adults, and few teachers. Informal studies indicate that few adults use this process in their daily lives. (The reader may ask himself how long it has been since, in his daily life outside of school, he has had to solve a problem like

$$\frac{3}{8} \div \frac{5}{7} .)$$

There seems to be an excellent case for considering this topic as a part of

[61] An example would be: True or false, $4 - 5 = -1$? In some curricula, "4" is identified with "−4", and the statement is called *true*; in other curricula, "4" is restricted to the system of natural numbers, within which $4 - 5$ has no answer, so the statement is *false*. Still other interpretations exist *for this same example* (e.g., an interpretation using equivalence classes of ordered pairs of natural numbers).

algebra, and not a part of *arithmetic*, and moving it from its present location (around grade 5) to a new location in grade 9.

All changes of this type—and they are very important changes, indeed—are hampered by the testing rationale that deals in terms of "a year's growth" or "scoring at the level of a typical 9th grader." Indeed, this rationale for interpreting tests seems to the present writer to be indefensible, since it imagines that education is a single race-track along which *all* students move. In fact, education is so multi-dimensional that many students never trace paths taken by other students, and their "progress" cannot be judged in terms of how far they have progressed along a path which they have not taken.

Differences *among different mathematics programs* are as great, in this sense, as the differences between students who have studied German vs. those who instead have studied Chinese. No common "test" will enable you to compare their progress. (Actually, criteria of a quite different sort *could* be devised *within each program*—such as the ability to read a newspaper, the ability to ask for directions and to understand the answer, etc. Such criteria would be very valuable, but they would not actually be "comparable" for the student of German vs. the student of Chinese.)

There is still another area of significant disagreement. Some people believe that there is a distinction between "knowledge in depth" vs. "superficial knowledge." Other observers deny this, and insist that observed behavior is the only valid criterion. This is usually interpreted to mean "observed responses on multiple-choice tests." In this connection, it is worth considering Gallileo's recanting before the pressure of the Inquisition, and (on June 21, 1633) reciting a statement, prepared by the Inquisition, in which he "abjured, cursed, and detested" his past errors. Did the Inquisition in fact "teach" Gallileo that his arguments in defense of the Copernican system were wrong? If we judge in terms of a narrow view of what Gallileo said, apparently they did. Are we justified in nonetheless inquiring into what Gallileo "actually believed"?

This, too, is not a quibble. Many critics maintain that a high proportion of today's students have come to "tell them [in school] what you know they want you to say," and not what you really believe. *There can be no graver danger to the healthy use of the human intellect.*

If this seems irrelevant to mathematics—and admittedly the situation is more critical in various other areas, such as literature, history, economics, political science, music, etc.—the reader can ask himself what the student is expected to believe concerning "how large one million is" or concerning the nature of mathematical proof and mathematical truth—or,

for that matter, on what it is that Euclidean geometry describes, or "what a *point* really is."

To turn to a simple and practical matter: do you, or do you not, want to ask the student to measure the classroom, and to observe how he goes about it: what devices (such as a ruler or a meter stick) he uses, how he handles them, etc.? Do you want to ask him, when he is done, whether his answer is correct? Do you want to confront him with another student who got a different answer, and see how he copes with this situation—whether he ignores it, becomes defensive, becomes aggressive, rechecks his procedure, studies the procedure of the other student, etc.? Do you want to ask him how he would find *exactly* the right answer?

14. **The Webster College Preservice Teacher Education Program.** The major effort of the "new mathematics" program is turning more and more to the question of teacher education: both the undergraduate college education of those who plan to become teachers, and the in-service (or other) continuing education of those who already are teachers. The Committee on the Undergraduate Program (CUPM) of the Mathematical Association of America (MAA)[62] has recommended that all prospective elementary school teachers study nine hours of specially-devised college mathematics, and that approximately 20 percent of all prospective elementary teachers take what amounts to an undergraduate major in mathematics while in college. The first university known to adopt this program—New Mexico State University at Las Cruces, New Mexico—was able to report an increase in the quality of the students electing to major in elementary education after the new program went into effect. Members of both the School of Education and the Mathematics Department report themselves well-pleased with the new program.

At the same time, it must be admitted that not all those who have reviewed the CUPM recommendations in detail agree that these recommendations hit the target squarely. Many feel that the kind of mathematical direction indicated by the Nuffield Project comes closer to what young children—and the teachers of young children—really need than the CUPM recommendations do.

Probably the most extensive undergraduate program in mathematics for prospective elementary school teachers is the program at Webster College, in Webster Groves, Missouri, created under the administrative leadership of the college president, Sister Jacqueline Grennan, S.L., and her predecessor, Sister Francetta Barberis, S.L., and designed by Katharine

[62] *Report on Eleven Conferences on the Training of Teachers of Elementary School Mathematics*, No. 13 (April 1966), Committee on the Undergraduate Program in Mathematics (CUPM), P.O. Box 1024, Berkeley, California 94701.

Kharas, retiring chairman of the Mathematics Department, and her successor, Richard Singer.

Webster College, being a small institution (800 students at present), elected to specialize in a small number of programs, and to bring to these both quality and originality. The College chose to abandon its elementary education major, and to undertake instead to prepare elementary school specialist teachers for the growing number of elementary schools that make use of some form of subject-matter specialization (whether through team teaching, semi-departmentalization, departmentalization, visiting specialist mathematics teachers, informal *ad hoc* arrangements among classroom teachers, or the use of resource teachers). Elementary specialist programs operate at Webster in several areas: various foreign language specialists, mathematics specialists, and physical science specialists, among others.

Distinctive features of the Webster program include:

a. In order to enter the program, students must have a strong high school mathematics background (which is in contrast to the nation-wide pattern of students electing elementary education);

b. The program consists of 30 credits of college-level mathematics —probably more than the average United States secondary teacher of mathematics has;

c. All courses are newly designed, and relate the college mathematics wherever possible to modern elementary school mathematics programs;

d. As at Nova, a large share of the responsibility has been shifted to the students themselves; for example, students undertake an extensive program of student teaching outside of and in addition to their formal college program, requesting faculty help only when they feel it is needed, but otherwise operating seminars without faculty involvement;

e. A student at Webster who pursues the maximum additional program of student teaching—and many do—will get practice in *every one* of the following:

Team teaching in a culturally-advantaged elementary school, intermediate grade level

Intermediate grade level specialist teaching in a culturally-disadvantaged area

Nursery school teaching experience

Experience teaching science and mathematics as a combined "unified" subject (intermediate grade level)

Some experience teaching primary-grade children

Experience teaching high school mathematics in a carefully-supervised "modern" program

Experience teaching short "seminar" courses to *college* students

Experience working with modern curriculum-revision projects as writers or trial teachers

Experience *teaching teachers* (in summer in-service programs).

Some students lengthen their total college program by one term or one year in order to include more "student teaching" of the various types; in any event, the "student teaching" program can begin in the freshman year and can continue for at least four years, *including summers.*

This program has been in operation for six years, and has been demonstrated to be viable and successful. So far as is known, it has not yet inspired similar programs in any other colleges.[63]

15. **The Webster College MAT Program.** Based on the experience with the undergraduate program just described, Katharine Kharas and Donald Cohen of Webster College have created a Master of Arts in Teaching degree that is *mathematically* somewhat similar to the undergraduate program, but fills an entirely different need. Superior practicing elementary school classroom teachers—usually with quite a few years of experience—pursue a 30-credit program of post-baccalaureate studies, *in mathematics,* leading to a "Master of Arts in Teaching" degree and a genuine competence as elementary school mathematics specialists. This program has already had a clearly visible impact—in the form of dramatically stronger mathematics programs—on at least two St. Louis area school systems. It has dramatic implications for other urban (and suburban) areas if it can be replicated elsewhere.[64] The effect of this program

[63]The "new mathematics," wherever it represents any real improvement in education, goes far beyond "mere mathematics." This is also the case with the Webster College program. The reform of the mathematics courses at Webster is important, and so is the large *amount* of mathematics that students study. So is the extent and variety of "student teaching" experiences, and the fact that students are given far more responsibility than formerly. But the Webster story goes beyond this: taking seriously Bruner's paradigm for "inductive" or "experiential" learning, Webster has tried everywhere to avoid *talking* about things wherever the students can learn more effectively by *doing* them. (Students, on their own, created a nursery school program for culturally deprived urban children and kept it in operation. This served as one of the models that influenced the designers of Project Head Start. Incidentally, it involved "politely educated" upper middle class white southern girls taking an apartment and living in a Negro urban slum in order to be closer to their nursery school pupils. Webster holds that education is not a game—education is for real!). In particular, the entire departmental arrangement of the college has been altered, *every course that seemed ineffective has been deleted,* and the traditional compartments into which college education is usually forced have been removed or relocated.

[64]Cf. Frank Peters and Robert LaRouche. "The New Math." *St. Louis Post-Dispatch,* Sunday, November 6, 1966, *Pictures* section, pp. 58 ff.

in the St. Louis area can be seen *by actually visiting elementary school classrooms.*

16. **The Chicago "Teacher-of-Teachers" Program.** The Chicago Public Schools have approximately 18,000 teachers in grades K-8 who are involved in the teaching of mathematics. Recognizing that in a system of this size, short-term programs for 35 or 50 teachers could have no noticeable effect on the child in the classroom, Associate Superintendent Evelyn Carlson and Mathematics Coordinator Bernice Antoine created a program, now in its fourth year of operation, to select around 600 specially competent teachers, provide them with intensive summer-workshops and Saturday workshops during the school year, and prepare them to conduct after-school workshops for the other Chicago teachers in their own local area. Both Patrick Suppes' project at Stanford University and the Madison Project have cooperated with Chicago in this program.

The quite evident success of the Chicago program has led to the development of similar programs elsewhere: John Huffman, Jack Price and Donald Hankins have created a similar program for San Diego County; other similar programs have been created in Los Angeles (by George Arbogast), in New York City (by George Grossman and Ella Huerstel Simpson), in Philadelphia (by Karl Kalman and Milton Goldberg), and in Corpus Christi, Texas (by Preston Kronkosky).

One interesting aspect of these programs is that many of the same people teach in most or all of them, thereby constituting what is, in effect, an operational "domestic teacher corps" that is already in existence at the present time. Hopefully, something more systematic may emerge from this effort—perhaps a university-based "domestic teacher corps" with a literally nation-wide area of operations.

17. **The Cambridge Conference on School Mathematics.** Who, if anyone, is planning for the future of mathematics education in the United States? Given our tradition of local control, plus the added problem of various schisms within the academic community, this becomes a difficult question.

If we look for a moment at the planning procedures of various organizations, it becomes clear that *no one appears to have the initiative.* This by itself is not entirely undesirable; few educators—and certainly not the present writer—would advocate (or even willingly tolerate) a monolithic federally-dictated educational program for the United States. What is more serious is that *there is no satisfactory arena for the discussion of serious planning.* No textbook publisher has an adequate in-house "research-and-development" capability, comparable with what Boeing, IBM, or Bell Telephone maintains (although some quite modest

progress in this direction is discernible at present). For many school systems, "local curriculum planning" may be more hypocrisy than accomplished fact, and usually comes down to nothing beyond the selection of one textbook from among a few available textbooks.

Here it might be added that this decision often seems to be carried by superficial criteria of little real relevance: the use of colors in the texts, or the physical durability of the actual book, when it is in fact the *program of study* that ought to be under scrutiny. About one percent of school expenses covers the total expenditure for all educational materials, including textbooks—consequently, were this to be increased three-fold, the result would be merely a two percent increase in school expenditures; total billing of the entire textbook industry for nursery school through graduate school amounts to about 600 million dollars annually (by contrast, the Viet Nam war costs two billion dollars every month). Other dubious criteria include: the use or non-use of certain *words* such as "borrow," "carry," "cancel," "transpose", (which are thought to be old-fashioned and evil) or "set," "union," "intersection," "inverse," "commutative," "associative," etc. (which are thought to be modern and virtuous), and the variety of pictures, allusions, etc. (even if they remain unrelated to what the child is actually doing).

Boeing Aircraft can invest millions of dollars in research and development ("R and D"), knowing that when it can deliver a superior aircraft it can find eager customers among the airlines of the world—you cannot wave your hands in a display of masterly salesmanship and persuade airlines to accept an aircraft that is unsafe or uneconomical to operate—in any event, not for long. A publisher does not enjoy the same assurance that a superior product will win recognition, with the result that publishers invest more heavily in sales personnel and hardly invest at all in an extensive "R and D" capability.

There is much more that might be said—and should be—concerning the publishers' role in the "new math revolution," but for the present it is worth noting that publishers do not possess a well-developed advance planning facility, that such planning as is done is aimed at the likelihood of securing adoptions rather than achieving a "breakthrough" to a genuinely new and improved curriculum, and that the publisher is handicapped by the unsophisticated criteria commonly used by textbook adoption committees.

If the school is not planning for the "program of the future," and the publisher is not, who is? Not state departments of education, in most cases—and for them to assume a major responsibility in this area would necessitate developing effective safeguards to protect local control (such safeguards undoubtedly *could* be devised).

Are schools of education planning for the "program of the future"? The answer has certainly been "no!"—as E. E. Moise has pointed out, SMSG should have been the creation of a school of education, but it was not. The same could be said of PSSC, BSCS, etc. Indeed, no major "new math" project yet has sprung up from any school of education —with the sole exception of the large array of effective projects that has grown up in the School of Education of the University of Illinois at Urbana. In fact, schools of education usually carry no significant budget item for "R and D" in specific curriculum areas. Within schools of education the word "research" has been taken to mean little more than "measurement and evaluation," and has ignored the actual domain of real research: *significant innovation*. That is one of the reasons why we may land a man on the moon before we succeed in changing the curriculum in our schools.

We stand now—perhaps—on the threshold of a new age: the federally-supported "Regional Laboratories" were invented as a new organizational entity to assume a significant share of the burden of innovation and advance planning. If in fact the Regional Laboratories are able to measure up to this task, we shall see a new interpretation of the meaning of "R and D" in the field of education.

Before writing off the past entirely, it must be admitted that several organizations have played important roles as serious forums for discussions of innovation and advanced planning. Probably the most noteworthy have been Educational Services, Incorporated,[65] of Newton, Massachusetts (a creation of MIT and Harvard, under the leadership of Jerrold Zacharias and Francis Friedman; ESI was formerly located in Watertown, Massachusetts), and the semi-annual "Directors Meetings" of the National Science Foundation. As an aid to further planning, ESI created the Cambridge Conference on School Mathematics, which has already produced a suggested outline for pre-college mathematics.[66] This report bears the appropriate admonition that it should by no means be regarded as a definitive blueprint. Since the summer of 1963, when *Goals* was written, the Cambridge Conference has conducted various feasibility studies, especially at the elementary school level, and during the summer of 1966 the Conference turned its attention to the question of teacher education; the resulting report on teacher education is now in press, and should appear soon.

[65] For information about this organization, which has influenced the curriculum revision movement perhaps more than any other organization, write to: Educational Services, Inc., 55 Chapel St., Newton, Massachusetts 02158.

[66] *Goals for School Mathematics*. The Report of the Cambridge Conference on School Mathematics. Boston: Houghton Mifflin Co., 1963.

For discussion of the Cambridge Conference, see (among others):

Irving Adler. "The Cambridge Report: Blueprint or Fantasy?" *The Mathematics Teacher* 59(3): 210-17; March 1966.

Marshall H. Stone. "Review of *Goals for School Mathematics: The Report of the Cambridge Conference on School Mathematics.*" *The Mathematics Teacher* 58(4): 353-60; April 1965.

Garrett R. Foster, Burt A. Kaufman and William M. Fitzgerald. *A First Step Towards the Implementation of the Cambridge Mathematics Curriculum in a K-12 Ungraded School.* Report submitted to the Commissioner of Education, U.S. Office of Education, in relation to Cooperative Research Project No. S-405. Florida State University, Tallahassee, Florida, 1966.

Perhaps forums for printed discussions are even more important than forums for face-to-face oral discussions. Probably the three most important forums for printed discussions during the past few years have been these:

The Arithmetic Teacher, a publication of the National Council of Teachers of Mathematics,[67] under the editorship of retiring editor Glenadine Gibb, and her successor, Marguerite Brydegaard (of San Diego State College). The May 1966 issue is devoted entirely to the best overview of school mathematics to appear in recent years (see particularly the article by John R. Mayor). The October 1966 and November 1966 issues give an outstanding view of "modern mathematics" at the classroom level, presented in what amounts to refined "lesson plans" for various "new mathematics" topics.

The Journal of Research in Science Teaching,[68] despite its name, deals equally with mathematics, and deserves to be better-known than it is.

Finally, perhaps the outstanding organization on the international scene is the Association of Teachers of Mathematics,[69] based in England. Its journal, *Mathematics Teaching*, surely publishes the most hard-hitting yet well-intended dissections and discussions that can be found in print. (Cf., for example, D. H. Wheeler's review of J. G. Wallace's book, *Concept Growth and the Education of the Child*, which appears on pages 66 and 67 of *Mathematics Teaching*, number 33, winter 1965.)

18. **The Report of Task Force No. 3, New York Public Schools.** Not content to cope with the problems of mere survival that confront the nation's largest school system, the late Joseph O. Loretan, Deputy

[67] Available from The National Council of Teachers of Mathematics, 1201 Sixteenth St., N.W., Washington, D. C. 20036.

[68] Available from John Wiley & Sons, Inc., 605 Third Avenue, New York, New York 10016.

[69] For information, write to Claude Birtwistle, Vine Street Chambers, Nelson, Lancashire, Great Britain.

Superintendent of the New York City Schools requested his mathematics coordinator, George Grossman, to prepare a program for moving the curriculum in the New York schools from where it stood in 1966 to a new curriculum resembling the projections of the Cambridge Conference. "Task Force No. 3" was created to carry out this remarkable undertaking, and—more remarkable yet—was able to produce a report for a multi-level mathematics program, pre-kindergarten through grade 12, that represents a considerable step towards "modernization" while at the same time maintaining relevance to the needs of the school children in the New York City population. For information, write to: George Grossman, Mathematics Coordinator, New York City Board of Education Annex, Room 201, 131 Livingston Street, Brooklyn, New York 11201.

The task of moving this curriculum from its present status as a report recorded on paper, to an operational status as a program of activities in the lives of children in New York schools, now lies ahead. This is an ambitious program, and if it can be accomplished it will be one more brick in the rebuilding and renewal of education in American cities.

19. **The "Academy of Mathematics."** We have considered earlier the relative absence of mathematical scholars in our schools. Dr. Loretan's successor, Acting Deputy Superintendent Helene M. Lloyd, of the New York City Schools, has proposed the creation of "academies" in various areas of knowledge. This proposal has not yet reached the stage of implementation, but the arguments that can be cited in its support are impressive. Dr. Lloyd's "academy of mathematics" would be an ongoing local activity of the New York City Schools, probing local community expertise both within the school, and from local universities or industries. Many school systems (University City, Missouri, is a case in point) have had composers-in-residence. There are presently proposals to have sculptors-in-residence. (The "jazz welfare" program conducted by Mercer Ellington in the New York City Schools suggests the potential value of this kind of effort, for example in providing success experiences and ego ideals for children who badly need both.) Is it foolish to imagine mathematicians-in-residence? This is only one of several roles that mathematicians might play in the schools—and, indeed, already are playing, as Cremin has remarked.[70]

[70] Lawrence A. Cremin, preface to the special edition on "American Intellectuals and the Schools." *Harvard Educational Review* 36(4): 391-93; Fall 1966. As another suggestive fact, Bruce Vogeli of Teachers College, Columbia University, who organized the tour of Russian schools by United States mathematicians and educators during the summer of 1966, has reported on the extensive participation of Russian mathematicians in direct work in the schools.

Consider the kinds of questions that face schools in the years ahead: Shall we replace Euclidean synthetic geometry with vector geometry? Or with projective geometry? Shall we use attribute blocks in the primary grades? How shall we arrange in-service education in mathematics for (say) the 20,000 teachers of grades K-6 in the New York City Schools? How can we plan *our own school program* in a way that really means something, and that is not a mere selection of some available text series? How can we constitute a sophisticated consumer of the publishers' wares, encouraging publishers to search more vigorously for the best possible *program*, instead of merely printing books and seeking adoptions?

In all of these matters, "resident mathematicians" and an "academy of mathematics" might be useful—provided it were arranged in an effective organizational way (which, among other requirements, would mean leaving scholars the amount of freedom they require, avoiding a premature determination of the role they are to play vis-a-vis other professionals, effectively focusing on *both* the child and the subject matter, and protecting against unnecessary inter-organizational resentments).[71]

The "academy" idea could mean freeing up—"unfreezing," in the words of Professor Alexander—the *entire* school curriculum. Consider the proposal that law be taught in our schools: would an "academy of law," or at least a carefully selected "lawyer-in-residence" not be essential to such a development if high quality standards are to be established?[72] (By "high quality" I do *not* mean "difficult." There appears to be no special correlation between the quality of experience *offered to* the student and the difficulty of the tasks that are *demanded of* the student.)

20. **The Increasing Internationalization of Mathematics.** Mathematics is, of course, one of the most international of all subjects. It plays the same role in one technologically-advanced society that it plays in any other. American schools and curriculum projects are coming increasingly to recognize this, and to take effective advantage of it, led by Howard Fehr and Bruce Vogeli (Teachers College, Columbia), Marshall Stone (University of Chicago), Peter Hilton (Cornell University), and Zoltan P. Dienes (Faculte des Sciences, Universite de Sherbrooke, Sherbrooke, Province of Quebec, Canada).

[71] Cf. the essay by David Hawkins, "Childhood and the Education of Intellectuals." *Harvard Educational Review* 36(4): 477-83; Fall 1966. Obviously, serious child study in the sense of Piaget (or ethology in the sense of Lorenz) would be an essential ingredient. Cf. the strong words of warning in Morris Kline's article, "Intellectuals and the Schools: A Case History." *Harvard Educational Review* 36(4): 505-11; Fall 1966.

[72] Cf. Paul A. Freund. "The Law and the Schools." *Harvard Educational Review* 36(4): 470-76; Fall 1966.

How Does It All Add Up?

We have looked very briefly at a few of the activities that deserve to be included in the phrase "new mathematics curricula." The phrase is unquestionably ill-chosen, yet (like many another misnomer) it has such broad currency that we have used it nonetheless; in fact, much of what is going on is perhaps not "new" (depending upon what *that* word means); more than merely mathematics is involved; and more than merely "curriculum" is at stake. Indeed, one is reminded again and again of such neo-MacLuhanisms ("the medium is the message") as: the classroom activity is the curriculum; the teaching strategy is the content; the objective-means-setting system contains its own evaluation; and so on.

Certainly if "new math" is to mean anything worthwhile, then it *does not* mean what most people think it does. It is *not* primarily a matter of "sets" and "binary numerals"—although, again, this is what people often mean when they say "the new math." *This meaning is shallow, unproductive, and misleading; it must be rejected.*

From the various pieces of "new math" that we have seen, can we draw any conclusions or make any generalizations? I propose to state a few that appeal to me, but drawing conclusions is, above all else, a do-it-yourself activity. The reader may read my conclusions if he chooses, but the important step is to become as familiar as possible with all of the various aspects of "new math," *and to draw your own conclusions.* Here are mine:

1. Some very promising directions for the future evolution of our schools are beginning to appear. For example: the Nuffield Project has discarded the classroom of forty children listening to one teacher, and replaced it by a mathematics laboratory with a dozen or so groups, of 3 children each, working on a dozen different and interesting tasks; at Stanford, at Urbana, and elsewhere, computer-assisted teaching machines are recording every response of every child to every question, analyzing the results, and using the analysis to revise the materials and to schedule every individual child through his own personal program of activities; the Nova-Carbondale program of Burt Kaufman has indicated that top-track students can do nearly the entire traditional

high school program as "homework," and can use *school* time to begin the study of college mathematics as early as grade eight. Programs initiated in Chicago, Los Angeles, San Diego, New York City, Philadelphia, St. Louis, and Washington, D.C. indicate that very significant improvements can be made in the mathematical diet assimilated by so-called "culturally-deprived" urban children.

2. Nonetheless, while some promising directions can now be identified, *very little has actually been accomplished.* All of the proposed changes—or at least, all of the really valuable ones—are difficult to implement. They require expenditure of money, they require extensive teacher education programs, they require further innovation in order to create a larger repertoire of teaching strategies and instructional materials, they require improved relations between schools and universities, and in some cases they even require a reconsideration of school architecture and of administrator-faculty relationships.[73] Perhaps above all, they require the *sense of commitment,* the determination, the optimism, the resourcefulness, the persistence, and the unwillingness to settle for too little, that one sees again and again in the life and work of distinguished educators, as revealed in the pages of Cremin's volume, *The Transformation of the School.*

3. Most discussions—and, apparently, far too much actual thinking—about "new mathematics" are harmfully superficial. The various words are used with so many different meanings that they have, in fact, none at all. The varieties of paradigms and assumptions are as numerous as the participants in the discussion, their differences are hard to reconcile, and—what is surely worse—these differences are left unnoticed and are not brought out into the clear light of forthright communication.

If there is one thing I wish to say to the reader, it is a plea to try to avoid meaningless and harmful superficiality. Before we say "we are using the new math" let us ask "what aspects of which versions of 'the new math'?" Before we ask "whether a program has been evaluated," let us raise the essential questions about what the program is seeking to accomplish, about whether we mean a "program" as it may appear on paper or some "program" as it may operate in actual classrooms, about what student population *and what teacher population* are involved

[73]One of the best examples of what this means can be seen in the remarkable Title III Project at Elk Grove, Illinois, which is directed by Mrs. Gloria Kinney, with the cooperation of Professor William Rogge, Mrs. Phyllis Farrel, and Mrs. Doris Machtinger, among others. At the time that this program was devised, the Superintendent was Dr. Roger Bardweli.

(and what sort of teacher education program has been used), about the kinds of observations of teachers that would convince us that this program was in fact being experienced by children, about the kinds of "learning" or "wisdom" or "knowledge" or "attitudes" or "feelings" or "abilities" we hope the children will acquire from their participation in the program, and about the kinds of observations of the students that would convince us that they had in fact obtained the profit that we hoped they would. Before we say "we are teaching 'sets'," we need to ask *which* of the various theories of "sets" we are using, exactly what sort of experiences with "sets" are the children getting, at what point are the children studying sets (and for what reason), and what purposes seem to be effectively achieved by this study? (Indeed, we had best also ask: what *alternatives* might instead be used?)

Every one of the following words and phrases is used in such different ways as to build confusion, rather than to minimize it:

the new math
discovery
sets
numeral systems
number systems
experience
concrete experience
meaningful
"use of the environment as a teaching tool"
abstract
structure
structural materials
achievement
evaluation
objectives
applications

and, in fact, many more.

4. Whether one looks at schools, or at universities, or at publishing houses, or even in several other possible places, one is led to conclude that the world of mathematics education is a culturally-deprived, newly-emerging (which means backward) area. It is not unpopulated, and some of the residents have achieved remarkable results, but the area is not one of the bright spots in American intellectual life. It is largely a no-man's-land, ventured into by an occasional administrator or psychologist (who usually finds the mathematical terrain mysterious and distasteful), ventured into by an occasional mathematician

or scientist (who is usually on a trip away from home), populated by an occasional genius (David Page, Burt Kaufman, Max Beberman, Geoffrey Matthews, Leonard Sealey, Francis Friedman, and Jack Easley come immediately to mind), but largely peopled by unsettled immigrants whose identity is defined by what they are *not:* they are not mathematicians, they are not psychologists, they are not scientists, they are not administrators and frequently they are not teachers.

More and more talented people are being attracted into the area, but even they are usually confronted by the fact that, like "newly-emerging" areas everywhere, the world of mathematics education is undercapitalized. Serious progress cannot be made—at least not easily—without some of the necessary tools of exploration: computers that record the response of every child to every question, modify the program accordingly, and individually schedule each child's work; videotape and film to record classroom behavior (even football has this equipment, and uses it very well indeed); adequate financial support to free innovation from the unworthy compromises of immediate commercial requirements; teamwork situations that can bring a significant force to bear on a worthwhile problem for an extended period of time; and so forth.

I do not for a moment wish to suggest that fencing in the area of mathematics education with an academic Chinese wall would constitute progress. Such an isolationist economy is the last thing this area needs! On the contrary, at the same time that this area builds its teams, acquires its computers and TV systems, and generally emerges from its gloomy past, it must also build very effective trade routes to unite it with mathematics and anthropology and curriculum design and physics and economics and school architecture and cognition and developmental psychology and educational history. The academic ghettos of the past do not provide the ideal model for the future—indeed, they do not provide even a minimally viable one.

5. This, above all: education is not a game. The present writer—and, most likely, the present reader—have livings to earn and families to support. Jobs in education may make this possible. Yet the schools do not in fact exist for our benefit; they serve a purpose as serious as the purpose of hospitals (and with no schools, or bad enough ones, there can be no hospitals, or only very bad ones); they serve a purpose as serious as armies and nuclear missiles (to be sure, this hardly seems to be recognized, but it is absolutely sure that man cannot fail to recognize it for many more years); they serve a purpose as serious as legislatures and courts (in this case the famous remark of Thomas

Jefferson remains accurate today and will continue to be accurate in the future); they serve a purpose at least as serious as whiskey, cosmetics, tobacco, detergents, floor wax, and automobiles.

We cannot apologize for a relentless impatience to improve education. We can never settle for "business as usual." We cannot merely write learned papers for the edification of our colleagues and the enhancement of our own professional image, or for the enlargement of our income. Our schools are not as good as they might be; they are not as good as they need to be; and it is our job to make them better. The verdict of our colleagues on us will count as nothing compared to the verdict of history on our civilization.

Bibliography

1. Background

For background on the "curriculum evolution" movement of the 1950's and 1960's, the references most commonly recommended are, in the view of the present writer, not in fact the most valuable. In their place, I would suggest the following:

Lawrence A. Cremin. *The Transformation of the School.* New York: Vintage Books, Random House, 1961.

This book provides an unsurpassed—indeed, unequaled—view of the historical background of this movement. Cremin writes history in such a way that its relevance to today's questions and tomorrow's decisions is made quite clear. Reading Cremin's *Transformation* is not a meaningless academic exercise; it is the surest road toward understanding the "new curricula."

Scientific American, Vol. 215, No. 3, September 1966.

This entire issue is devoted to computers; it has a twofold relevance to education: in the first place, it shows how our technology and our society are changing (so that it indicates the *demands* that will be made on education in the next few years); in the second place, it shows how computers are being used to study education and learning (so that it suggests a major direction of modern educational research) and how computers are being used to program individual children in school (so that it gives a view of what schools may come to look like in the years ahead). Thus the various articles of this issue deal with both the *goals* of education (or, at least, some of them) and also the *means* that may come to be used in pursuing these goals.

2. Films and Booklets

The following films and booklets are especially recommended. They give a view of an important portion of "the new mathematics curricula" that is all too often overlooked:

FILM: *I Do . . . And I Understand.* A 14 minute black and white 16 mm sound motion picture film, available from: Mr. S. Titheradge, Manager, New Print Department, Sound Services, Ltd., Wilton Crescent, Merton Park, London, S.W.19, England.

BOOKLET: *I Do . . . And I Understand.* Available from the Nuffield Mathematics Project, 12 Upper Belgrave St., London, S.W.1, England.

This booklet was designed to accompany the film of the same name, listed on p. 75. Nonetheless, *the booklet can be read separately*, and deserves to be read, with or without the film.

FILM: *Maths Alive*. Available from The Foundation Library, Brooklands House, Weybridge, Surrey, England.

BOOKLET: *A Look Ahead*. Available from the Nuffield Mathematics Project, 12 Upper Belgrave St., London, S.W.1, England.

3. Lists and Surveys

Two excellent "overviews" of the present (1966-67) state of affairs concerning the mathematics curriculum are presented in the first two papers below, by Gerald Rising and by John R. Mayor. The Rising article is especially recommended; it should not be missed!

Gerald R. Rising. "Elementary School Mathematics Curriculum Revision —The State of (the) Art." *New York State Mathematics Teachers Journal* 16(3): 90-109; June 1966.

John R. Mayor. "Issues and Directions." *The Arithmetic Teacher* 13(5): 349-54; May 1966.

Robert B. Davis. "Mathematics." Chapter 6 in the volume *New Curriculum Developments*. Glenys Unruh, editor. Available from: Association for Supervision and Curriculum Development, 1201 Sixteenth St., N.W., Washington, D.C. 20036 (1965).

J. David Lockard. *Report of the International Clearinghouse on Science and Mathematics Curricular Developments 1966*. Available from: Science Teaching Center, University of Maryland, College Park, Maryland 20740.

UNESCO (United Nations Educational, Scientific, and Cultural Organization). *University Instruction in Mathematics. A Comparative Survey of Curricula and Methods in the Universities of Czechoslovakia, Federal Republic of Germany, France, Japan, Poland, United Kingdom, U.S.A., U.S.S.R.* Prepared by the International Commission of Mathematical Instruction (1966).

Despite its title, this survey deals also with pre-college mathematical curricula. It should go without saying that pre-college and college curricula *must be considered simultaneously*. To do otherwise would be to confess no serious interest in the education of the individual student in actual practice.

National Science Foundation (an agency of the United States Government). *Course and Curriculum Improvement Projects. Mathematics, Science, [and] Engineering. Elementary School, Secondary School, College, and University*. (September 1966.) Available from the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402 (NSF 66-22).

This volume is an absolute must for everyone interested in curriculum.

ESI Quarterly Report, Spring/Summer 1966. Available from: Educational Services, Incorporated, 55 Chapel Street, Newton, Massachusetts 02158.

This is not an extensive or comprehensive survey, but it does an unsurpassed job of giving the *flavor* of the "new curriculum" projects.

Howard Fehr *et al. New Thinking in School Mathematics* (May 1961). The official United States version of the Report of the Royaumont Seminar on New Thinking in Mathematical Education. Prepared for the OEEC (Organization for European Economic Co-operation), Office for Scientific and Technical Personnel. Available from OEEC Mission, Publications Office, Suite 1223, 1346 Connecticut Avenue, N.W., Washington, D. C. 20036.

4. Periodicals

Again, using the personal judgment of the present writer, the four most valuable journals concerned with "the new mathematics" appear to be:

The Arithmetic Teacher. Marguerite Brydegaard, editor. Available from: The National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington, D.C. 20036.

Journal of Research in Science Teaching. J. Stanley Marshall, editor. Available from: John Wiley & Sons, Inc., 605 Third Avenue, New York, New York 10016.

Mathematics Teaching. Claude Birtwistle, editor. Available from: The Association of Teachers of Mathematics, Vine Street Chambers, Nelson, Lancashire, England.

The Mathematics Teacher. Irvin H. Brune, editor. Available from: The National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington, D.C. 20036.

Of these four, the present writer (and most of his colleagues) have found the English journal *Mathematics Teaching* and the United States journal *The Arithmetic Teacher* to be of special value: we recommend them both highly. If any extra difficulty is involved in obtaining an English journal, the journal itself is well worth the effort.

It appears likely that some new journals will be created in this area in the next few years. They may be worth watching for! (An international journal is now being considered by Oxford University Press, and the question of an appropriate journal has been put to the Cambridge Conference, to give two examples. There is also some discussion of starting a branch of the Association of Teachers of Mathematics in the United States.)

5. Research, Philosophy, Innovation and Evaluation

J. A. Easley, Jr. "The Natural Sciences and Educational Research—A Comparison." *The High School Journal* 50(1): 39-50; October 1966.

This article, and the one by Gardner Quarton (listed below), are two of the most thoughtful that have appeared, by two of the most profound writers

to concern themselves with "new curricula," especially where research, philosophy, innovation and evaluation are concerned. The articles by Levin, by Morrison, and by Sealey also command attention.

Harry Levin. "Reading Research: What, Why, and for Whom?" *Elementary English* (February 1966). pp. 138-47.

Philip Morrison. "Tensions of Purpose." *ESI Quarterly Report*, Spring/Summer 1966. pp. 67-70. (Available from Educational Services, Incorporated, 55 Chapel Street, Newton, Massachusetts 02158.)

Gardner Quarton. "Evaluating New Science Materials: Thoughts on Methods and Goals." *ESI Quarterly Report*, Spring/Summer 1966. pp. 77-79. (Available from Educational Services, Incorporated.)

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